ROMANIAN MATHEMATICAL MAGAZINE

Let a, b, c be non − negative real numbers such that

$a+b+c=ab+bc+ac$. Prove that :

$$\frac{a}{\sqrt{7a+2}}+\frac{b}{\sqrt{7b+2}}+\frac{c}{\sqrt{7c+2}}\geq 1$$

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Let $a=\frac{1}{x}, b=\frac{1}{y}, c=\frac{1}{z}$, WLOG we can assume that x & z are on the same side of unity.

$$\Rightarrow x+y+z=xy+xz+yz.$$

Thus it's sufficient to show that :

$$\frac{1}{\sqrt{2x^2+7x}}+\frac{1}{\sqrt{2y^2+7y}}+\frac{1}{\sqrt{2z^2+7z}}\geq 1.$$

Since we have that $f(u)=\frac{1}{\sqrt{2u^2+7u}}$ is a convex function thus we can conclude that

$$\frac{1}{\sqrt{2x^2+7x}}+\frac{1}{\sqrt{2z^2+7z}}\overset{\text{Jensen's}}{\geq}\frac{2}{\sqrt{2\left(\frac{x+z}{2}\right)^2+7\left(\frac{x+z}{2}\right)}}.$$

Let assume that $x+z=2t$ and It's given that

$$\begin{cases} x+y+z=xy+xz+yz \\ (x-1)(z-1)\geq 0 \Rightarrow xz\geq 2t-1 \end{cases} \Rightarrow \begin{cases} y=\frac{2t-xz}{2t-1}\leq\frac{1}{2t-1} & \text{for } \frac{1}{2}<t. \\ y=\frac{xz-2t}{1-2t}\overset{AM-GM}{\leq}\frac{t^2-2t}{1-2t} & \text{for } 0\leq t<\frac{1}{2}. \end{cases}$$

And since that the function

$f(u)=\frac{1}{\sqrt{2u^2+7u}}$ is a decreasing function we conclude that

Case (I):

$$\frac{2}{\sqrt{2t^2+7t}}+\frac{1}{\sqrt{2y^2+7y}}\geq\frac{2}{\sqrt{2t^2+7t}}+\frac{1}{\sqrt{2\left(\frac{1}{2t-1}\right)^2+7\left(\frac{1}{2t-1}\right)}}\geq 1.$$

Case (II):

$$\frac{2}{\sqrt{2t^2+7t}}+\frac{1}{\sqrt{2y^2+7y}}\geq\frac{2}{\sqrt{2t^2+7t}}+\frac{1}{\sqrt{2\left(\frac{t^2-2t}{1-2t}\right)^2+7\left(\frac{t^2-2t}{1-2t}\right)}}\geq 1.$$

With equality at $t=1\ or\ t\to 0.5\ \Rightarrow (a,b,c)=(1,1,1)\ or\ (a,b,c)$
$=(2,2,0)$ **and their permutations.**