ROMANIAN MATHEMATICAL MAGAZINE

**Let $a,b,c\geq 0$ such that $ab+bc+ca=3$. Prove that**

$$\frac{1}{\sqrt{5a+bc}}+\frac{1}{\sqrt{5b+ca}}+\frac{1}{\sqrt{5c+ab}}\geq\frac{\sqrt{6}}{2}$$

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Let $p:=a+b+c,\ \ q:=ab+bc+ca=3,\ \ r:=abc\overset{AM-GM}{\leq}1$. By Hölder's inequality, we have

$$\left(\sum_{cyc}\frac{1}{\sqrt{5a+bc}}\right)^2\left(\sum_{cyc}(5a+bc)(b+c)^3(2a+b+c)^3\right)\geq\left(\sum_{cyc}(b+c)(2a+b+c)\right)^3=8\left(p^2+3\right)^3.$$

So it suffices to prove that

$$16\left(p^2+3\right)^3\geq 3\sum_{cyc}(5a+bc)(b+c)^3(a+p)^3,$$

which, after expanding and simplifying becomes,

$$f(r)=7p^6-45p^5+144p^4-135p^3+432p^2-2835p+432+3\left(p^5+5p^4+12p^3-90p^2+45p+315\right)r-3\left(4p^2-35p+15\right)r^2\geq 0.$$

We have $\quad f'(r)=3\left[p^5+5p^4+12p^3-90p^2+45p+315-2\left(4p^2-35p+15\right)r\right]$.

▪If $4p^2-35p+15\leq 0$, by using AM − GM inequality, we have

$$f'(r)\geq 3\left(p^5+5p^4+12p^3+45p+315-90p^2\right)\geq 3\left(4\sqrt[4]{5p^4.12p^3.45p.315}-90p^2\right)>0.$$

▪If $4p^2-35p+15\geq 0$, since $r\leq 1$ and by using AM − GM inequality, we have

$$f'(r)\geq 3\left(p^5+5p^4+12p^3-98p^2+115p+285\right)\geq 3\left(4\sqrt[4]{5p^4.12p^3.115p.285}-98p^2\right)>0,$$

then $f$ is increasing. And since $p\geq\sqrt{3q}=3$ then we have two cases :

▪If $p\geq 2\sqrt{3}$, we have

$$f(r)\geq f(0)=\left(p-2\sqrt{3}\right)\left[\left(7p+14\sqrt{3}-45\right)p^4+\left(228-90\sqrt{3}\right)p^3+\left(456\sqrt{3}-675\right)p^2+\left(3168-1350\sqrt{3}\right)p+6336\sqrt{3}-10935\right]+54\left(712-405\sqrt{3}\right)\geq 0.$$

▪If $3\leq p\leq 2\sqrt{3}$, by Schur's inequality, we have $r\geq\dfrac{p\left(4q-p^2\right)}{9}=r'$, then $f(r)\geq f(r')$ with

$$f(r')=\frac{1}{27}(p-3)\left\{\left(2\sqrt{3}-p\right)\left[13p^6+\left(49+26\sqrt{3}\right)p^5+\left(98\sqrt{3}+33\right)p^4+\left(924+66\sqrt{3}\right)p^3+\left(1848\sqrt{3}-3159\right)p^2+\left(11583-6318\sqrt{3}\right)p+23166\sqrt{3}-50787\right]+101574\sqrt{3}-142884\right\}\geq 0.$$

So the proof is complete. Equality holds iff $a=b=c=1$.