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Let $a,b,c\geq 0$ such that $ab+bc+ca=3$. Prove that
$$\sqrt{a+1}+\sqrt{b+1}+\sqrt{c+1}\geq 3\sqrt{2}$$

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WLOG, we assume that $0\leq a\leq b\leq c$. Since $3=ab+bc+ca\geq 3a^2$, then $0\leq a\leq 1$.

We have $a^2+3=(a+b)(a+c)\leq\left(\frac{2a+b+c}{2}\right)^2$, then $b+c\geq 2\left(\sqrt{a^2+3}-a\right)$.

Let $t:=\sqrt{a^2+3}-a\in\left[1,\sqrt{3}\right]$. We have $b+c\geq 2t$, $a=\frac{3-t^2}{2t}$, and

$$\left(\sqrt{b+1}+\sqrt{c+1}\right)^2=b+c+2+2\sqrt{b+c+bc+1}=b+c+2+2\sqrt{(b+c)(1-a)+4}$$
$$\geq 2t+2+2\sqrt{2t\left(1-\frac{3-t^2}{2t}\right)+4}=4(t+1).$$

Then

$$\sqrt{a+1}+\sqrt{b+1}+\sqrt{c+1}\geq\sqrt{a+1}+2\sqrt{t+1}=\sqrt{\frac{3-t^2}{2t}+1}+2\sqrt{t+1}\overset{?}{\geq}3\sqrt{2}$$
$$\Leftrightarrow 4\sqrt{\left(\frac{3-t^2}{2t}+1\right)(t+1)}\geq 13-4t-\frac{3-t^2}{2t}.$$

If $13\leq 4t+\frac{3-t^2}{2t}$, the last inequality is true.

Otherwise, after squaring, the last inequality is equivalent to

$$-1+28t-62t^2+44t^3-9t^4\geq 0\Leftrightarrow (t-1)^2[t(25-9t)+t-1]\geq 0,$$

which is true for all $t\in\left[1,\sqrt{3}\right]$. So the proof is complete.

Equality holds iff $a=b=c=1$.