ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c\geq 0, ab+bc+ca>0$. Prove that:

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}\leq\frac{6(a^3+b^3+c^3)+9abc}{(a+b+c)^2}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$(a+b+c)^2\left(\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}\right)=\sum_{cyc}\left[(b+c)^2+2a(b+c)+a^2\right].\frac{b^2+c^2}{b+c}$$

$$=\sum_{cyc}(b+c)\left(b^2+c^2\right)+2\sum_{cyc}a\left(b^2+c^2\right)+\sum_{cyc}a^2.\frac{(b+c)^2-2bc}{b+c}$$

$$=2\sum_{cyc}a^3+3\sum_{cyc}a^2(b+c)+\sum_{cyc}\left(a^2(b+c)-2abc.\frac{a}{b+c}\right)$$

$$\overset{Nesbitt}{\leq}2\sum_{cyc}a^3+4\sum_{cyc}a^2(b+c)-2abc.\frac{3}{2}\overset{Schur}{\leq}2\sum_{cyc}a^3+4\left(\sum_{cyc}a^3+3abc\right)-3abc$$

$$=6\left(a^3+b^3+c^3\right)+9abc,$$

as desired. Equality holds iff $(a=b=c>0)$ and $(a=b>0, c=0)$ and permutation.