ROMANIAN MATHEMATICAL MAGAZINE

Let $a, b, c \geq 0, ab + bc + ca > 0$. Prove that :

$$\frac{a^2 + b^2 - c^2}{a + b} + \frac{b^2 + c^2 - a^2}{b + c} + \frac{c^2 + a^2 - b^2}{c + a} \leq \frac{a + b + c}{2}$$

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Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Multiplying the both sides of the desired inequality by $2(a + b)(b + c)(c + a)$, we get the equivalent inequality

$$2\sum_{cyc}(b^2 + c^2 - a^2)(a + b)(a + c) \leq (a + b + c)(a + b)(b + c)(c + a),$$

which, after expanding and simplifying, is equivalent to

$$\sum_{cyc} a^3(b + c) + 2\sum_{cyc}(bc)^2 \leq 2\sum_{cyc} a^4 + 2abc\sum_{cyc} a$$

$$\Leftrightarrow \ 0 \leq 2\sum_{cyc} a^2(a - b)(a - c) + \sum_{cyc} bc(b - c)^2,$$

which is true by fourth degree Schur's inequality.

Equality holds iff $(a = b = c > 0)$ and $(a = b > 0, c = 0)$ and permutation.