ROMANIAN MATHEMATICAL MAGAZINE

Let $a, b, c \geq 0, ab+bc+ca > 0$. Prove that :

$$\frac{a(b+c)}{ab+ac+2bc}+\frac{b(c+a)}{bc+ba+2ca}+\frac{c(a+b)}{ca+cb+2ab}\leq\frac{(a+b+c)^2}{2(ab+bc+ca)}$$

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$$2(ab+bc+ca)\sum_{cyc}\frac{a(b+c)}{ab+ac+2bc}=\sum_{cyc}\left(a(b+c)+\frac{a^2(b+c)^2}{ab+ac+2bc}\right)\leq$$

$$\overset{CBS}{\leq} 2\sum_{cyc}bc+\sum_{cyc}a^2\left(\frac{b^2}{ab+bc}+\frac{c^2}{ac+bc}\right)=2\sum_{cyc}bc+\sum_{cyc}\left(\frac{a^2b}{a+c}+\frac{ca^2}{a+b}\right)=$$

$$=2\sum_{cyc}bc+\sum_{cyc}\left(\frac{c^2a}{c+b}+\frac{ab^2}{b+c}\right)=2\sum_{cyc}bc+\sum_{cyc}\frac{a(b^2+c^2)}{b+c}=$$

$$=2\sum_{cyc}bc+\sum_{cyc}\left(a(b+c)-\frac{2abc}{b+c}\right)\overset{CBS}{\leq}4\sum_{cyc}bc-2abc.\frac{9}{\sum_{cyc}(b+c)}=$$

$$=2\sum_{cyc}bc+\left(2\sum_{cyc}bc-\frac{9abc}{a+b+c}\right)\overset{Schur}{\leq}2\sum_{cyc}bc+\sum_{cyc}a^2=(a+b+c)^2,$$

as desired. Equality holds iff $(a=b=c>0)$ and $(a=0, b=c>0)$ and permutation.