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Let $a,b,c\ge 0, ab+bc+ca>0$ and $a+b+c=3$. Prove that :

$$\frac{a}{\sqrt{bc}+3}+\frac{b}{\sqrt{ca}+3}+\frac{c}{\sqrt{ab}+3}\le\frac{9}{4(ab+bc+ca)}$$

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$$4(ab+bc+ca)\sum_{cyc}\frac{a}{\sqrt{bc}+3}=(ab+bc+ca)\sum_{cyc}\frac{4a}{\sqrt{bc}+a+b+c}\le$$

$$\overset{CBS}{\underbrace{\le}}(ab+bc+ca)\sum_{cyc}a\left(\frac{1}{b+c}+\frac{1}{a+\sqrt{bc}}\right)=$$

$$=\sum_{cyc}\frac{a(ab+bc+ca)}{b+c}+(ab+bc+ca)\sum_{cyc}\frac{a}{a+\sqrt{bc}}=$$

$$=\sum_{cyc}a^2+\sum_{cyc}\frac{abc}{b+c}+\left(3-\sum_{cyc}\frac{\sqrt{bc}}{a+\sqrt{bc}}\right).\sum_{cyc}bc\le$$

$$\overset{\substack{HM-GM\\CBS}}{\underbrace{\le}}\sum_{cyc}a^2+\sum_{cyc}\frac{a\sqrt{bc}}{2}+\left(3-\frac{\left(\sum_{cyc}\sqrt{bc}\right)^2}{\sum_{cyc}\sqrt{bc}\left(a+\sqrt{bc}\right)}\right).\sum_{cyc}bc=$$

$$=\sum_{cyc}a^2+\frac{1}{2}\sum_{cyc}a\sqrt{bc}+\left(2-\frac{\sum_{cyc}a\sqrt{bc}}{\sum_{cyc}bc+\sum_{cyc}a\sqrt{bc}}\right).\sum_{cyc}bc\le$$

$$\le\sum_{cyc}a^2+\frac{1}{2}\sum_{cyc}a\sqrt{bc}+\left(2-\frac{\sum_{cyc}a\sqrt{bc}}{2\sum_{cyc}bc}\right).\sum_{cyc}bc=(a+b+c)^2=9,$$

Equality holds iff $(a=b=c>0)$ and $(a=0,b=c>0)$ and permutation.