ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c \geq 0$ such that : $ab + bc + ca > 0$. Prove that :

$$\frac{a^2(b+c)}{b^2+bc+c^2}+\frac{b^2(c+a)}{c^2+ca+a^2}+\frac{c^2(a+b)}{a^2+ab+b^2}\geq\frac{2(a^2+b^2+c^2)}{a+b+c}$$

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Solution 1 by Soumava Chakraborty-Kolkata-India

$\boxed{Case\ 1}$ Exactly one variable $= 0$ and WLOG we may assume $a = 0$ and

$$\text{then : } \frac{a^2(b+c)}{b^2+bc+c^2}+\frac{b^2(c+a)}{c^2+ca+a^2}+\frac{c^2(a+b)}{a^2+ab+b^2}=\frac{b^4}{b^2c}+\frac{c^4}{c^2b}$$

$$\overset{\text{Bergstrom}}{\geq}\frac{(b^2+c^2)^2}{bc(b+c)}\overset{A-G}{\geq}\frac{2bc(b^2+c^2)}{bc(b+c)}=\frac{2(a^2+b^2+c^2)}{a+b+c}$$

$$\therefore \frac{a^2(b+c)}{b^2+bc+c^2}+\frac{b^2(c+a)}{c^2+ca+a^2}+\frac{c^2(a+b)}{a^2+ab+b^2}\geq\frac{2(a^2+b^2+c^2)}{a+b+c}$$

$\boxed{Case\ 2}$ $a, b, c > 0$

Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \rightarrow (1) \Rightarrow a = s - x, b = s - y, c = s - z$

$\therefore abc = r^2 s \rightarrow (2)$ and such substitutions $\Rightarrow \sum_{cyc} ab = \sum_{cyc}(s-x)(s-y)$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \rightarrow (3), \sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab$$

$$\overset{\text{via (1) and (3)}}{=} s^2 - 2(4Rr + r^2) \Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \rightarrow (4),$$

$$\sum_{cyc} a^2b^2 = \left(\sum_{cyc} ab\right)^2 - 2abc\left(\sum_{cyc} a\right) \overset{\text{via (1),(2) and (3)}}{=} (4Rr + r^2)^2 - 2r^2 s.s$$

$$\Rightarrow \sum_{cyc} a^2b^2 = r^2\left((4R+r)^2 - 2s^2\right) \rightarrow (5),$$

$$\sum_{cyc} a^3 = \left(\sum_{cyc} a\right)^3 - 3(a+b)(b+c)(c+a) \overset{\text{via (1)}}{=} s^3 - 3.4Rrs$$

$$\Rightarrow \sum_{cyc} a^3 = s(s^2 - 12Rr) \to (6) \text{ and } \sum_{cyc} a^4 = \left(\sum_{cyc} a^2\right)^2 - 2\sum_{cyc} a^2b^2 \overset{\text{via } (4) \text{ and } (5)}{=}$$

$$(s^2 - 8Rr - 2r^2)^2 - 2r^2((4R + r)^2 - 2s^2)$$

$$\Rightarrow \sum_{cyc} a^4 = (s^2 - 8Rr - 2r^2)^2 - 2r^2((4R + r)^2 - 2s^2) \to (7)$$

$$\text{Now, } (a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2)$$

$$= \left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right) + abc\sum_{cyc} a^3 + 2abc\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 6a^2b^2c^2$$

$$+ \left(\sum_{cyc} ab\right)^3 - 3r^2s.4Rrs \overset{\text{via } (1),(2),(3),(4),(5) \text{ and } (6)}{=}$$

$$r^2(s^2 - 8Rr - 2r^2)((4R + r)^2 - 2s^2) + r^2s^2(s^2 - 12Rr) + 2r^2s^2(4Rr + r^2)$$

$$-6r^4s^2 + (4Rr + r^2)^3 - 12Rr^3s^2$$

$$\therefore \boxed{(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \overset{(*)}{=} r^2(s^2(4R + r)^2 - s^4 - r(4R + r)^3)}$$

$$\text{Also, } \left(\sum_{cyc} a\right)\left(\sum_{cyc} \left(a^2(b + c)(c^2 + ca + a^2)(a^2 + ab + b^2)\right)\right)$$

$$= a^2(b + c)\left(\sum_{cyc} a^2b^2 + (a^3 + abc)\left(\sum_{cyc} a\right)\right)$$

$$= \left(\sum_{cyc} a^2b^2 + abc\left(\sum_{cyc} a\right)\right)\left(\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a\right)^2 - 3abc\left(\sum_{cyc} a\right)\right)$$

$$+ \left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^4\right)\left(\sum_{cyc} a\right)^2 - abc\left(\sum_{cyc} a^3\right)\left(\sum_{cyc} a\right)^2$$

$$\overset{\text{via } (1),(2),(3),(5),(6) \text{ and } (7)}{=} \left(r^2((4R + r)^2 - 2s^2) + r^2s^2\right)\left((4Rr + r^2)s^2 - 3r^2s^2\right)$$

$$+(4Rr + r^2)\left((s^2 - 8Rr - 2r^2)^2 - 2r^2((4R + r)^2 - 2s^2)\right)s^2 - r^2s^4(s^2 - 12Rr)$$

$$\Rightarrow \boxed{\left(\sum_{cyc} a\right)\left(\sum_{cyc} \left(a^2(b + c)(c^2 + ca + a^2)(a^2 + ab + b^2)\right)\right)}$$

$$\overset{(**)}{=} 2rs^2(2Rs^4 - rs^2(32R^2 + 4Rr - r^2) + 6Rr^2(4R + r)^2)$$

$$\therefore (*), (**) \Rightarrow \frac{a^2(b + c)}{b^2 + bc + c^2} + \frac{b^2(c + a)}{c^2 + ca + a^2} + \frac{c^2(a + b)}{a^2 + ab + b^2} \geq \frac{2(a^2 + b^2 + c^2)}{a + b + c}$$

$$\Leftrightarrow \frac{2rs^2\left(2Rs^4 - rs^2\left(32R^2 + 4Rr - r^2\right) + 6Rr^2(4R+r)^2\right)}{r^2\left(s^2(4R+r)^2 - s^4 - r(4R+r)^3\right)} \geq 2\left(s^2 - 8Rr - 2r^2\right)$$

$$\Leftrightarrow (2R+r)s^6 - rs^4\left(48R^2 + 20Rr + 2r^2\right) + r^2s^2\left(288R^3 + 192R^2r + 42Rr^2 + 3r^3\right)$$

$$-2r^3(4R+r)^4 \overset{(\bullet)}{\geq} 0 \text{ and } \because (2R+r)\left(s^2 - 16Rr + 5r^2\right)^3$$

$$+r\left(48R^2 - 2Rr - 17r^2\right)\left(s^2 - 16Rr + 5r^2\right)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{ in order to prove } (\bullet),$$

$$\text{it suffices to prove : LHS of } (\bullet) \geq (2R+r)\left(s^2 - 16Rr + 5r^2\right)^3 +$$

$$r\left(48R^2 - 2Rr - 17r^2\right)\left(s^2 - 16Rr + 5r^2\right)^2$$

$$\Leftrightarrow \left(144R^3 - 80R^2r - 76Rr^2 + 49r^3\right)s^2 \overset{(\bullet\bullet)}{\geq}$$

$$r\left(2304R^4 - 2048R^3r - 600R^2r^2 + 876Rr^3 - 149r^4\right)$$

$$\text{Now, LHS of } (\bullet\bullet) \overset{\text{Gerretsen}}{\geq} \left(144R^3 - 80R^2r - 76Rr^2 + 49r^3\right)\binom{16Rr}{-5r^2}$$

$$\overset{?}{\geq} r\left(2304R^4 - 2048R^3r - 600R^2r^2 + 876Rr^3 - 149r^4\right)$$

$$\Leftrightarrow 2t^3 - 9t^2 + 12t - 4 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (2t-1)(t-2)^2 \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \text{ is true}$$

$$\therefore \frac{\left(\sum_{\text{cyc}} a\right)\left(\sum_{\text{cyc}}\left(a^2(b+c)\left(c^2+ca+a^2\right)\left(a^2+ab+b^2\right)\right)\right)}{\left(a^2+ab+b^2\right)\left(b^2+bc+c^2\right)\left(c^2+ca+a^2\right)} \geq 2\sum_{\text{ccy}} a^2$$

$$\Rightarrow \frac{a^2(b+c)}{b^2+bc+c^2} + \frac{b^2(c+a)}{c^2+ca+a^2} + \frac{c^2(a+b)}{a^2+ab+b^2} \geq \frac{2\left(a^2+b^2+c^2\right)}{a+b+c}$$

$$\therefore \text{combining both cases}, \frac{a^2(b+c)}{b^2+bc+c^2} + \frac{b^2(c+a)}{c^2+ca+a^2} + \frac{c^2(a+b)}{a^2+ab+b^2}$$

$$\geq \frac{2\left(a^2+b^2+c^2\right)}{a+b+c} \ \forall a,b,c \geq 0 \,|\, ab+bc+ca > 0,$$

$$'' ='' \text{ iff } (a=0, b=c\neq 0) \text{ or } (b=0, c=a\neq 0) \text{ or } (c=0, a=b\neq 0) \text{ or}$$

$$(a=b=c\neq 0) \text{ (QED)}$$

Solution 2 by Nguyen Van Canh-Vietnam

WLOG, we assume that $a+b+c=1$. **Let** $q = ab+bc+ca \leq \frac{(a+b+c)^2}{3} = \frac{1}{3}, r = abc.$

We have:

$$\frac{a^2(b+c)}{b^2+bc+c^2} + \frac{b^2(c+a)}{c^2+ca+a^2} + \frac{c^2(a+b)}{a^2+ab+b^2} \geq 2(a^2+b^2+c^2);$$

$$\Leftrightarrow \frac{\sum[a^2(b+c)(c^2+ca+a^2)(a^2+ab+b^2)]}{(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)} \geq 2(a^2+b^2+c^2);$$

$$\Leftrightarrow \frac{\sum ab\left(a^5+b^5\right) + \sum a^2b^2\left(a^3+b^3\right) + \sum a^3b^3(a+b) + 2abc\sum a^4 + 2abc\sum ab\left(a^2+b^2\right) + 4a^2b^2c^2\sum a + 2abc\sum a^2b^2}{\sum a^2b^2(a^2+b^2) + \sum a^3b^3 + abc\sum a^3 + 2abc\sum ab(a+b) + 3(abc)^2}$$

$$\geq 2(a^2+b^2+c^2);$$

$$\Leftrightarrow \frac{\sum a^5\sum ab + abc\sum a^4 + \sum a^2b^2\sum a^3 + 3(abc)^2\sum a + \sum a^3b^3\sum a + abc\sum a^2b^2 + 2abc\sum ab(a^2+b^2)}{\sum a^2b^2\sum a^2 + \sum a^3b^3 + abc\sum a^3 + 2abc\sum ab(a+b)}$$

$$\geq 2\left(a^2+b^2+c^2\right);$$

$$\Leftrightarrow \frac{q(1-5q+5q^2+5r-5qr)+r(1-4q+2q^2+4r)+(q^2-2r)(1-3q+3r)+3r^2+q^3-3qr+3r^2+r(q^2-2r)+2r(q-2q^2-r)}{(q^2-2r)(1-2q)+q^3-3qr+3r^2+r(1-3q+3r)+2r(q-3r)}$$
$$\geq 2(1-2q):$$

$$\Leftrightarrow \frac{3q^3-4q^2+q+(6q-3q^2-1)r}{q^2-q^3-r} \geq 2(1-2q);$$
$$\Leftrightarrow 3q^3-4q^2+q+(6q-3q^2-1)r \geq 2(1-2q)(q^2-q^3-r);$$
$$\Leftrightarrow (1+2q-3q^2)r-4q^4+9q^3-6q^2+q \geq 0;$$
$$\Leftrightarrow (1-q)(1+3q)r-4q^4+9q^3-6q^2+q \geq 0 \quad (*)$$

By Schur's Inequality we have: $r \geq \max\left\{0, \frac{4q-1}{9}\right\}$.

- **If** $0 < q \leq \frac{1}{4}$ **then** $r \geq 0$**, we have:**

$$(1-q)(1+3q)r-4q^4+9q^3-6q^2+q \geq -4q^4+9q^3-6q^2+q$$
$$= q(1-4q)(q-1)^2 \geq 0 \ \ (\text{true}) \Rightarrow (*) \ \text{true}.$$

- **If** $\frac{1}{4} \leq q \leq \frac{1}{3}$ **then** $r \geq \frac{4q-1}{9}$**, we have:**

$$(1-q)(1+3q)r-4q^4+9q^3-6q^2+q$$
$$\geq \frac{(1-q)(1+3q)(4q-1)}{9}-4q^4+9q^3-6q^2+q$$
$$= -4\,q^4+\frac{23\,q^3}{3}-\frac{43\,q^2}{9}+\frac{11\,q}{9}-\frac{1}{9}$$
$$= \frac{1}{9}(1-q)(4q-1)(3q-1)^2 \geq 0 \ (\text{true}) \Rightarrow (*) \ \text{true}.$$

Proved. Equality $\Leftrightarrow a=b=c$ **or** $a=0, b=c\neq 0$ **or** $b=0, a=c\neq 0$ **or** $c=0, a=b\neq 0$.