ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c\ge 0, ab+bc+ca+abc=4.$ Prove that :

$$\frac{1}{2a+2b+c}+\frac{1}{2b+2c+a}+\frac{1}{2c+2a+b}\le\frac{21}{10a+10b+10c+5}$$

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Let $p:=a+b+c, q:=ab+bc+ca, r:=abc.$ We have $q+r=4,$ and

$$\sum_{cyc}\frac{1}{2b+2c+a}=\sum_{cyc}\frac{1}{2p-a}=\frac{\sum_{cyc}(2p-b)(2p-c)}{(2p-a)(2p-b)(2p-c)}=\frac{8p^2+q}{4p^3+2pq-r}=$$

$$\overset{q=4-r}{=}\frac{8p^2+4-r}{4p^3+8p-(2p+1)r}\overset{?}{\le}\frac{21}{10p+5}\Leftrightarrow r\overset{?}{\le}\frac{p^3-10p^2+32p-5}{4(2p+1)}=r_0.\ (1)$$

By AM − GM inequality, we have

$$4=q+r\ge 3\sqrt[3]{r^2}+r\Rightarrow r\le 1\ \ \&\ \ p\ge\sqrt{3q}=\sqrt{3(4-r)}\ge 3.$$

If $p\ge 7,$ we have : $\dfrac{p^3-10p^2+32p-5}{4(2p+1)}=1+\dfrac{(p-3)(p^2-7p+3)}{4(2p+1)}\ge 1\ge r.$

If $3\le p\le 7$: Suppose that $r>r_0.$

Using the following known identity, we have

$$0\le(a-b)^2(b-c)^2(c-a)^2=p^2q^2-4q^3+18pqr-4p^3r-27r^2=$$

$$=p^2(4-r)^2-4(4-r)^3+18p(4-r)r-4p^3r-27r^2=$$

$$=16p^2-256-(4p^3+8p^2-72p-192)r-(75+18p-p^2)r^2+4r^3=f(r).$$

We have

$$f''(r)=-2(75+18p-p^2)+24r=-126-24(1-r)-2p(18-p)<0\Rightarrow f'\downarrow$$

$$\Rightarrow f'(r)<f'(r_0)=-4p^3-8p^2+72p+192-2(75+18p-p^2)r_0+12{r_0}^2=$$

$$=-\frac{(p-3)\left[(p-3)(2343+802p+423p^2+192p^3-7p^4)+7560\right]}{4(2p+1)^3}\overset{3\le p\le 7}{\le}0\Rightarrow f\downarrow$$

$$\Rightarrow f(r)<f(r_0)=16p^2-256-(4p^3+8p^2-72p-192)r_0-(75+18p-p^2){r_0}^2+4{r_0}^3=$$

$$= -(p-3)^2(p-4)^2\left(69+43p+137p^2+293p^3+127p^4-3p^5\right) \overbrace{\leq}^{3\leq p\leq 7} 0,$$

which contradicts $f(r) \geq 0$, so (1) is true, and the proof is complete.

Equality holds iff $(a=b=c=1)$ and $\left(a=b=\dfrac{1}{2}, c=3\right)$ and its permutations.