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If $a_k > 0 \ (k = 1, 2, \dots, n)$ *and* $s = \sum_{k=1}^{n} a_k$, then :

$$\sum_{k=1}^{n} a_k^3 \geq \frac{1}{n(n-1)}\left(\sum_{k=1}^{n} a_k \cdot \sqrt{s - a_k}\right)^2$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{n(n-1)}\left(\sum_{k=1}^{n} a_k \cdot \sqrt{s - a_k}\right)^2 \overset{\text{CBS}}{\leq} \frac{1}{n(n-1)}\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n}(s - a_k)\right)$$

$$= \frac{1}{n(n-1)}\left(\sum_{k=1}^{n} a_k^2\right)\left(n\sum_{k=1}^{n} a_k - \sum_{k=1}^{n} a_k\right) = \frac{n-1}{n(n-1)}\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} a_k\right)$$

$$\Rightarrow \frac{1}{n(n-1)}\left(\sum_{k=1}^{n} a_k \cdot \sqrt{s - a_k}\right)^2 \leq \frac{1}{n}\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} a_k\right) \to (1)$$

$$\text{Now, } \sum_{k=1}^{n} a_k^3 \overset{\text{Chebyshev}}{\geq} \frac{1}{n}\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} a_k\right) \overset{\text{via (1)}}{\geq} \frac{1}{n(n-1)}\left(\sum_{k=1}^{n} a_k \cdot \sqrt{s - a_k}\right)^2,$$

$$'' ='' \text{ iff } a_1 = a_2 = \cdots = a_n \text{ (QED)}$$