ROMANIAN MATHEMATICAL MAGAZINE

If $x_k > 0$ $(k = 1, 2, \dots, n)$, then :

$$\frac{1}{n}\sum_{cyc}\sqrt{1+\frac{6}{x_1+x_2+x_3}+\frac{3}{x_1x_2+x_2x_3+x_3x_1}}-\frac{n}{\sum_{k=1}^{n}x_k}\geq 1$$

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$$\frac{1}{n}\sum_{cyc}\sqrt{1+\frac{6}{x_1+x_2+x_3}+\frac{3}{x_1x_2+x_2x_3+x_3x_1}}-\frac{n}{\sum_{k=1}^{n}x_k}$$

$$\geq \frac{1}{n}\sum_{cyc}\sqrt{1+\frac{6}{x_1+x_2+x_3}+\frac{9}{(x_1+x_2+x_3)^2}}-\frac{n}{\sum_{k=1}^{n}x_k}$$

$$=\frac{1}{n}\sum_{cyc}\sqrt{\frac{(x_1+x_2+x_3)^2+6(x_1+x_2+x_3)+9}{(x_1+x_2+x_3)^2}}-\frac{n}{\sum_{k=1}^{n}x_k}$$

$$=\frac{1}{n}\sum_{cyc}\sqrt{\left(\frac{x_1+x_2+x_3+3}{x_1+x_2+x_3}\right)^2}-\frac{n}{\sum_{k=1}^{n}x_k}=\frac{1}{n}\sum_{cyc}\left(1+\frac{3}{x_1+x_2+x_3}\right)-\frac{n}{\sum_{k=1}^{n}x_k}$$

$$=\frac{1}{n}.n+\frac{3}{n}\sum_{cyc}\frac{1}{x_1+x_2+x_3}-\frac{n}{\sum_{k=1}^{n}x_k}\overset{\text{Bergstrom}}{\geq}1+\frac{3}{n}.\frac{n^2}{3\sum_{k=1}^{n}x_k}-\frac{n}{\sum_{k=1}^{n}x_k}=1$$

$$\therefore \frac{1}{n}\sum_{cyc}\sqrt{1+\frac{6}{x_1+x_2+x_3}+\frac{3}{x_1x_2+x_2x_3+x_3x_1}}-\frac{n}{\sum_{k=1}^{n}x_k}\geq 1$$

$\forall x_k > 0$ $(k = 1, 2, \dots, n)$, $'' = ''$ iff $x_1 = x_2 = \dots = x_n$ (QED)