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If $a_k > 0(k = 1,2,\dots,n)$ and $s = \sum_{k=1}^{n} a_k$, then:

$$\left(\frac{n-1}{n}\right)^2 \sum_{k=1}^{n} a_k \cdot \sum_{k=1}^{n} \frac{a_k}{(s-a_k)^2} \ge 1$$

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Solution by Marian Ursărescu – Romania

$$\sum_{k=1}^{n} a_k \cdot \sum_{k=1}^{n} \frac{a_k}{(s-a_k)^2} \ge \frac{n^2}{(n-1)^2} \Leftrightarrow \sum_{k=1}^{n} \frac{a_k^2}{(s-a_k)^2} + \sum_{k=1}^{n} \frac{a_k(s-a_k)}{(s-a_k)^2} \ge \frac{n^2}{(n-1)^2} \Leftrightarrow$$

$$\sum_{k=1}^{n} \left(\frac{a_k}{s-a_k}\right)^2 + \sum_{k=1}^{n} \frac{a_k}{s-a_k} \ge \frac{n^2}{(n-1)^2} \quad (1)$$

But $\sum_{k=1}^{n} \frac{a_k}{s-a_k} \ge \frac{n}{n-1}$ **(2)**

From Cauchy $n \cdot \sum_{k=1}^{n} \left(\frac{a_k}{s-a_k}\right)^2 \ge \left(\sum_{k=1}^{n} \frac{a_k}{s-a_k}\right)^2$

$$\sum_{k=1}^{n} \left(\frac{a_k}{s-a_k}\right)^2 \ge \frac{1}{n}\left(\sum_{k=1}^{n} \frac{a_k}{s-a_k}\right)^2 \overset{(2)}{\ge} \frac{1}{n} \cdot \frac{n^2}{(n-1)^2} = \frac{n}{(n-1)^2} \quad (3)$$

From (1) + (2) + (3) :

$$\sum_{k=1}^{n} a_k \cdot \sum_{k=1}^{n} \frac{a_k}{(s-a_k)^2} \ge \frac{n}{n-1} + \frac{n}{(n+1)^2} = \frac{n^2-n+n}{(n-1)^2} = \frac{n^2}{(n-1)^2}$$