ROMANIAN MATHEMATICAL MAGAZINE

If $x_i > 0 (i = 1, 2, \dots, n)$ and $k \in \mathbb{N}^*$, then prove that:

$$2\sum_{i=1}^{n} x_i^{2k+1} \ge \sum_{cyclic} x_1x_2\left(x_1^{2k-1} + x_2^{2k-1}\right) \ge \sum_{cyc} x_1^2x_2^2\left(x_1^{2k-3} + x_2^{2k-3}\right) \ge \cdots \ge$$

$$\ge \sum_{cyclic} x_1^kx_2^k(x_1 + x_2)$$

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For any $a, b \in \mathbb{R}_+$ and $p \in \mathbb{N}^*$, we have the following inequality:

$$a^{p+1} + b^{p+1} \ge ab(a^{p-1} + b^{p-1}) \Leftrightarrow a^p(a-b) - b^p(a-b) =$$

$$= (a-b)^2(a^{p-1} + \cdots + b^{p-1}) \ge 0$$

Furthermore, we have $a^{2p+1} + b^{2p+1} \ge ab(a^{2p-1} + b^{2p-1}) \ge$

$$\ge a^2b^2(a^{2p-3} + b^{2p-3}) \ge \cdots \ge a^pb^p(a+b)$$

Using this result k times, we obtain the desired inequality

$$2\sum_{i=1}^{n} x_i^{2k+1} = \sum_{cyc}\left(x_1^{2k+1} + x_2^{2k+1}\right) \ge \sum_{cyc} x_1x_2\left(x_1^{2k-1} + x_2^{2k-1}\right) \ge$$

$$\ge \sum_{cyc} x_1^2x_2^2\left(x_1^{2k-3} + x_2^{2k-3}\right) \ge \cdots \ge \sum_{cyc} x_1^kx_2^k(x_1 + x_2).$$

Equality holds for $x_1 = x_2 = \cdots = x_n$.