FOUR PROOFS OF INEQUALITY $\binom{2m}{m} \geq 2^m, \forall m \in \mathbb{Z}^+$

Math note by:

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Abstract. In this math note, I would like to elaborate some kinds of proof of the inequality $\binom{2m}{m} \geq 2^m$ over $m \in \mathbb{Z}^+$. So far, I have found 4 types of proof: by direct algebraic computation, by using induction, applying Vandermonde and Newton Binomial identities, and combinatorial proof. Although only these 4 proofs are written here, I believe that there will be many other interesting methods and creative ideas, as an idiom says "All roads lead to Rome". More suggestions and developments are welcome.

MAIN RESULTS

PROOF 1: Direct Algebraic Computation

Clearly $\frac{m+k}{k} \geq 2$ for all $m \in \mathbb{Z}^+$ and $k = 1,2, \dots \dots, m$. Then,

$$\binom{2m}{m} = \frac{(2m)!}{m!\,m!} = \prod_{k=1}^{m} \left(\frac{m+k}{k}\right) \geq \prod_{k=1}^{m} 2 = 2^m.$$

PROOF 2: By Induction Principle

For $m \in \mathbb{Z}^+$, let $P(m)$ states that $\binom{2m}{m} \geq 2^m$. Note that $P(1)$ is true because $\binom{2m}{m} = 2^m = 2$ when $m = 1$. Assume that $P(i)$ is true for some $i \in \mathbb{Z}^+$, then $\binom{2i}{i} \geq 2^i$. Observe that

$$\frac{\binom{2i+2}{i+1}}{\binom{2i}{i}} = \frac{\frac{(2i+2)!}{(i+1)!\,(i+1)!}}{\frac{(2i)!}{i!\,i!}} = \frac{2(2i+1)}{i+1} \geq 2$$

then

$$\binom{2i+2}{i+1} \geq 2\binom{2i}{i} \geq 2 \cdot 2^i = 2^{i+1}$$

so $P(i+1)$ is also true. By induction, $P(m)$ is true for every positive integers m, i.e.

$$\binom{2m}{m} \geq 2^m, \forall m \in \mathbb{Z}^+.$$

PROOF 3: Vandermonde and Binomial theorem

Recall Vandermonde identity [1] : For all $m, n, r \in \mathbb{Z}^+$ then

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \dots\dots + \binom{m}{r}\binom{n}{0} = \binom{m+n}{r} \quad \dots\dots\dots (*)$$

While binomial expansion theorem says [2] : for all $n \in \mathbb{Z}^+$ and $x, y \in \mathbb{R}$ we have

$$(x+y)^n = \binom{n}{n} x^n y^0 + \binom{n}{n-1} x^{n-1} y^1 + \dots\dots + \binom{n}{1} x^1 y^{n-1} + \binom{n}{0} x^0 y^n$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \quad \dots\dots\dots\dots (**)$$

Setting $m, n, r \coloneqq m$ to equation $(*)$, we get

$$\binom{2m}{m} = \sum_{k=0}^{m} \binom{m}{k}\binom{m}{m-k} = \sum_{k=0}^{m} \binom{m}{k}^2 = \binom{m}{0}^2 + \binom{m}{1}^2 + \dots\dots + \binom{m}{m}^2.$$

Besides that, setting $x, y \coloneqq 1$ and $n \coloneqq m$ to equation $(**)$ give us

$$2^m = \sum_{k=0}^{m} \binom{m}{k} = \binom{m}{0} + \binom{m}{1} + \dots\dots + \binom{m}{m}.$$

We also know that $\binom{m}{k}^2 \geq \binom{m}{k}$ for all $k = 0,1,2, \dots\dots, m$. Therefore,

$$\binom{2m}{m} = \sum_{k=0}^{m} \binom{m}{k}^2 \geq \sum_{k=0}^{m} \binom{m}{k} = 2^m.$$

PROOF 4: A Combinatorial Proof

Let $S = \{a_1, a_2, \dots \dots, a_{2m}\}$ be set of $2m$ distinct objects, with $m \in \mathbb{Z}^+$. We partition this set into m disjoint 2-element subsets as follows.

$$S_1 = \{a_1, a_2\}, \quad S_2 = \{a_3, a_4\}, \quad S_3 = \{a_5, a_6\}, \dots \dots \dots, S_m = \{a_{2m-1}, a_{2m}\}.$$

Observe that the number of ways to choose m random objects from S without any constraint, that is $\binom{2m}{m}$, must be greater than or equal to the number of ways to choose m objects from S for which exactly one element in each S_i are taken, that is 2^m. Therefore, $\binom{2m}{m} \geq 2^m$.

REFERENCES

[1]. Vandermonde identity; Wikipedia The Free Encyclopedia; url: https://en.wikipedia.org/wiki/Vandermonde%27s_identity

[2]. Binomial theorem; Wikipedia The Free Encyclopedia; url: https://en.wikipedia.org/wiki/Binomial_theorem