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Prove that in *all* triangle ABC holds :

$$\sum_{cyc}(1-\cos A-\cos 2A-\cos(B-C))^2=\left(\frac{s^2-4Rr-r^2}{R^2}\right)^2-\left(\frac{s^2+4Rr+r^2}{2R^2}\right)^2$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}(1-\cos A-\cos 2A-\cos(B-C))^2$$

$$=\sum_{cyc}\left(1-\cos 2A+(\cos(B+C)-\cos(B-C))\right)^2=\sum_{cyc}\left(2\sin^2 A-2\sin B\sin C\right)^2$$

$$=\sum_{cyc}\left(\frac{2a^2}{4R^2}-\frac{2bc}{4R^2}\right)^2=\frac{1}{4R^4}\sum_{cyc}(a^4+b^2c^2-2a^2bc)$$

$$=\frac{1}{4R^4}\left(\left(\sum_{cyc}a^4+2\sum_{cyc}b^2c^2\right)-\left(\sum_{cyc}b^2c^2+2abc\sum_{cyc}a\right)\right)$$

$$=\frac{1}{4R^4}\left(\left(\sum_{cyc}a^2\right)^2-\left(\sum_{cyc}ab\right)^2\right)=\frac{4\left(s^2-4Rr-r^2\right)^2}{4R^4}-\frac{\left(s^2+4Rr+r^2\right)^2}{4R^4}$$

$$=\left(\frac{s^2-4Rr-r^2}{R^2}\right)^2-\left(\frac{s^2+4Rr+r^2}{2R^2}\right)^2\ \text{(QED)}$$