ROMANIAN MATHEMATICAL MAGAZINE

**Let ABC be $a$ triangle $a$nd J denote the midpoint of GH.**
**Prove th$a$t the center of the Euler's circle lies on the incircle if $a$nd only if**

$$IJ=\frac{\sqrt{2}}{3}IH$$

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$$HI^2=4R^2+4Rr+3r^2-s^2\to(1),\ GH^2=\frac{4}{9}\left(9R^2-\sum_{cyc}a^2\right)\to(2),$$

$$GI^2=\frac{1}{9}(s^2-16Rr+5r^2)\to(3)$$

$\because$ center of the Euler circle lies on the incircle, $\therefore NI=r$

$$(N\to\text{center of Euler circle})\Rightarrow\frac{2F.OI^2}{abc}=r\Rightarrow\frac{2rs.R(R-2r)}{4Rrs}=r\Rightarrow R=4r\to(4)$$

$$\text{Now, in }\Delta\,HIG,\ IJ\text{ is }a\text{ median}\Rightarrow 4IJ^2=2HI^2+2GI^2-GH^2\overset{?}{=}\frac{8}{9}HI^2\overset{\text{via }(1),(2),(3)}{\Leftrightarrow}$$

$$\frac{10}{9}(4R^2+4Rr+3r^2-s^2)\overset{?}{=}\frac{4}{9}\left(9R^2-\sum_{cyc}a^2\right)-\frac{2}{9}(s^2-16Rr+5r^2)$$

$$\Leftrightarrow 10(4R^2+4Rr+3r^2-s^2)\overset{?}{=}4\left(9R^2-2(s^2-4Rr-r^2)\right)-2(s^2-16Rr+5r^2)$$

$$\Leftrightarrow R^2-6Rr+8r^2\overset{?}{=}0\Leftrightarrow(R-2r)(R-4r)\overset{?}{=}0\overset{\text{via }(4)}{\Leftrightarrow}2r(R-4r)\overset{?}{=}0$$

$$\to\text{true via }(4)\ \therefore 4IJ^2=\frac{8}{9}HI^2\Rightarrow IJ=\frac{\sqrt{2}}{3}IH\ (QED)$$