ROMANIAN MATHEMATICAL MAGAZINE

In *all* non − isosceles Δ ABC, the following identity is true :

$$\sum_{cyc}\frac{\sin^4 A\sin\frac{A}{2}}{\sin\frac{A-B}{2}\sin\frac{A-C}{2}}=\frac{r(3s^2-r^2-4Rr)}{4R^3}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$b-c=4R\sin\frac{A}{2}\sin\frac{B-C}{2}\Rightarrow\sin\frac{B-C}{2}=\frac{b-c}{4R\sin\frac{A}{2}}\text{ and analogs}$$

$$\therefore\frac{4R^3}{r}.\sum_{cyc}\frac{\sin^4 A\sin\frac{A}{2}}{\sin\frac{A-B}{2}\sin\frac{A-C}{2}}=\frac{4R^3}{r}.\sum_{cyc}\frac{\sin^4 A\sin\frac{A}{2}}{\frac{a-b}{4R\sin\frac{C}{2}}\cdot\frac{a-c}{4R\sin\frac{B}{2}}}$$

$$=\frac{4R^3.16R^2}{r}.\sum_{cyc}\frac{\sin^4 A\left(\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)}{(a-b)(a-c)}=$$

$$=\frac{4R^3.16R^2}{r}.\sum_{cyc}\frac{\sin^4 A\left(\frac{r}{4R}\right)}{(a-b)(a-c)}=\sum_{cyc}\frac{(2R\sin A)^4}{(a-b)(a-c)}=$$

$$=\frac{a^4(b-c)}{(a-b)(a-c)(b-c)}+\frac{b^4(c-a)}{(b-a)(b-c)(c-a)}+\frac{c^4(a-b)}{(c-a)(c-b)(a-b)}$$

$$=\frac{a^4(b-c)+b^4(c-a)+c^4(a-b)}{(a-b)(a-c)(b-c)}=\frac{a^4(b-c)+(b^4c-bc^4)-a(b^4-c^4)}{(a-b)(a-c)(b-c)}$$

$$=\frac{(b-c)\left((a^4-ab^3)+(b^3c-ab^2c)+(b^2c^2-abc^2)-(ac^3-bc^3)\right)}{(a-b)(a-c)(b-c)}$$

$$=\frac{(b-c)(a-b)\left((a^3-c^3)+b(a^2-c^2)+b^2(a-c)\right)}{(a-b)(a-c)(b-c)}$$

$$=\frac{(b-c)(a-b)(a-c)(a^2+c^2+ca+ab+bc+b^2)}{(a-b)(a-c)(b-c)}=\sum_{cyc}a^2+\sum_{cyc}ab$$

$$=2(s^2-4Rr-r^2)+s^2+4Rr+r^2=3s^2-r^2-4Rr$$

$$\therefore\sum_{cyc}\frac{\sin^4 A\sin\frac{A}{2}}{\sin\frac{A-B}{2}\sin\frac{A-C}{2}}=\frac{r(3s^2-r^2-4Rr)}{4R^3}\ \text{(QED)}$$