Let ABCD be the convex quadrilateral where the angle $\widehat{BAD} = \frac{\pi}{2}, \hat{C} \in \left(0, \frac{\pi}{2}\right)$ with $AD = CD = BC = a > 0$. $AE \perp BD, E \in (BD)$ and p is the semiperimeter of ABCD. Prove that:

$$\left(\sqrt{AC.\cos\hat{C}} + \sqrt{AE.\left|\sin(\hat{B}+\hat{D})\right|}\right)^2 < a\sqrt{2} + \sqrt{\frac{a(2p-3a)}{2}}$$

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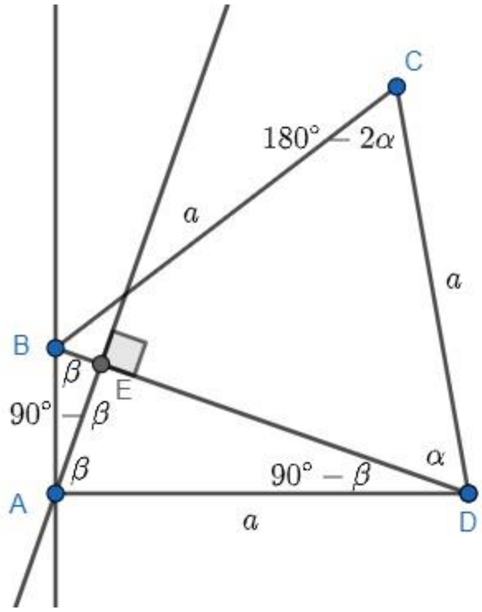

$$BD^2 = a^2 + a^2 - 2a^2\cos(180° - 2\alpha) = 4a^2\cos^2\alpha \Rightarrow BD \overset{(1)}{=} 2a\cos\alpha$$
$$(\because 180° - 2\alpha > 0 \Rightarrow 0 < \alpha < 90° \Rightarrow \cos\alpha > 0) \text{ and } AB^2 = BD^2 - AD^2$$
$$= 4a^2\cos^2\alpha - a^2 \Rightarrow AB \overset{(2)}{=} a.\sqrt{4\cos^2\alpha - 1} \text{ and } \therefore \Delta ABD \sim \Delta EAD \therefore \frac{BD}{AD} = \frac{AB}{AE}$$
$$\overset{\text{via (1) and (2)}}{\Rightarrow} \frac{2a\cos\alpha}{a} = \frac{a.\sqrt{4\cos^2\alpha - 1}}{AE} \Rightarrow AE \overset{(*)}{=} \frac{a}{2}.\frac{\sqrt{4\cos^2\alpha - 1}}{\cos\alpha}$$
Now, via Ptolemy's inequality, $AC.2a\cos\alpha < a.a.\sqrt{4\cos^2\alpha - 1} + a^2$
$$\Rightarrow AC \overset{(**)}{<} \frac{a}{2}.\frac{1 + \sqrt{4\cos^2\alpha - 1}}{\cos\alpha}$$
$$\left(\sqrt{AC.\cos\hat{C}} + \sqrt{AE.\left|\sin(\hat{B}+\hat{D})\right|}\right)^2 \overset{CBS}{\leq} (AC + AE)\left(\cos\hat{C} + \left|\sin(\hat{B}+\hat{D})\right|\right)$$
$$\overset{\text{via (*) and (**)}}{<} \frac{a\left(1 + 2\sqrt{4\cos^2\alpha - 1}\right)}{2\cos\alpha}.\left(-\cos 2\alpha + |\sin(90° + 2\alpha)|\right)$$
$$= \frac{a\left(1 + 2\sqrt{4\cos^2\alpha - 1}\right)(-2\cos 2\alpha)}{2\cos\alpha}$$
$$(\because 180° - 2\alpha < 90° \Rightarrow 2\alpha > 90° \Rightarrow \cos 2\alpha < 0) \overset{?}{<} a\sqrt{2}$$
$$\Leftrightarrow \boxed{\sqrt{2}\cos\alpha \underset{(\bullet)}{\overset{?}{>}} \left(1 + 2\sqrt{4\cos^2\alpha - 1}\right)(-\cos 2\alpha)}$$

$$\text{Let }\sqrt{4\cos^2\alpha-1}=t\ \mathit{and}\ \because\ \alpha>45^\circ\Rightarrow\cos\alpha<\frac{1}{\sqrt{2}}\Rightarrow\sqrt{4\cos^2\alpha-1}<1$$

$$\Rightarrow 0<t<1\ \mathit{and}\ \cos^2\alpha=\frac{t+1}{4}\Rightarrow\cos\alpha=\frac{\sqrt{t+1}}{2}\ \therefore(\bullet)\Leftrightarrow$$

$$\sqrt{2}.\frac{\sqrt{t+1}}{2}>\left(1-\frac{t+1}{2}\right)(1+2t)\Leftrightarrow\sqrt{\frac{t+1}{2}}>\left(\frac{1-t}{2}\right)(1+2t)$$

$$\Leftrightarrow\frac{t+1}{2}>\frac{(1-t)^2(1+2t)^2}{4}\ (\because(1-t),t>0)\Leftrightarrow 4t^4-4t^3-3t^2-1<0$$

$$\Leftrightarrow 4t^3(t-1)-3t^2-1<0\rightarrow\text{true}\ \because\ 0<t<1\Rightarrow(\bullet)\text{ is true}$$

$$\therefore\left(\sqrt{AC.\cos\hat{C}}+\sqrt{AE.\left|\sin(\hat{B}+\hat{D})\right|}\right)^2<a\sqrt{2}<a\sqrt{2}+\sqrt{\frac{a(2p-3a)}{2}}\ (\text{QED})$$