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Let the quadrilateral ABCD circumscribe a circle of radius r and let A′B′C′D′ be the quadrilateral whose vertices are the points of contact of the sides of the quadrilateral ABCD with the circle. Prove that:

$$\sqrt[4]{r}\left(\frac{1}{\sqrt[4]{AA'}}+\frac{1}{\sqrt[4]{BB'}}+\frac{1}{\sqrt[4]{CC'}}+\frac{1}{\sqrt[4]{DD'}}\right)\ge\frac{64}{\frac{\sin\frac{A+B}{2}\sin\frac{B+C}{2}\sin\frac{C+A}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2}}+12}$$

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Solution by Soumava Chakraborty-Kolkata-India

Let $AA'=AD'=e, BB'=BA'=f, CC'=CB'=g$, $DD'=DC'=h$ and with reference to article "Calculations concerning the Tangent Lengths and Tangency Chords of a Tangential Quadrilateral" by Martin Josefsson, published in "Forum Geometricum", Volume 10 (2010) 119 – 130, we get:

$$r=\sqrt{\frac{\theta}{\xi}}\ (\xi=e+f+g+h \text{ and } \theta=efg+fgh+ghe+hef),$$

$$\sin\frac{A}{2}=\sqrt{\frac{\theta}{E}},\sin\frac{B}{2}=\sqrt{\frac{\theta}{F}},\sin\frac{C}{2}=\sqrt{\frac{\theta}{G}},\sin\frac{D}{2}=\sqrt{\frac{\theta}{H}}, \text{where}$$

$$E=(e+f)(e+g)(e+h), F=(f+e)(f+g)(f+h),$$
$$G=(g+e)(g+f)(g+h) \text{ and } H=(h+e)(h+f)(h+g)$$

$$\text{and using }\cos\frac{A}{2}=\sqrt{1-\sin^2\frac{A}{2}}\text{ and analogs, we get: }\cos\frac{A}{2}=e.\sqrt{\frac{\xi}{E}},$$

$$\cos\frac{B}{2}=f.\sqrt{\frac{\xi}{F}},\cos\frac{C}{2}=g.\sqrt{\frac{\xi}{G}}\therefore\sin\frac{A+B}{2}=\sin\frac{A}{2}\cos\frac{B}{2}+\cos\frac{A}{2}\sin\frac{B}{2}$$

$$=\sqrt{\frac{\theta}{E}}.f.\sqrt{\frac{\xi}{F}}+e.\sqrt{\frac{\xi}{E}}.\sqrt{\frac{\theta}{F}}\Rightarrow\sin\frac{A+B}{2}=\sqrt{\frac{\theta\xi}{EF}}(e+f)\text{ and analogously,}$$

$$\sin\frac{B+C}{2}=\sqrt{\frac{\theta\xi}{FG}}(f+g)\text{ and }\sin\frac{C+A}{2}=\sqrt{\frac{\theta\xi}{GE}}(g+e)\therefore\frac{\sin\frac{A+B}{2}\sin\frac{B+C}{2}\sin\frac{C+A}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2}}$$

$$=\frac{\left(\sqrt{\frac{\theta\xi}{EF}}(e+f)\right)\left(\sqrt{\frac{\theta\xi}{FG}}(f+g)\right)\left(\sqrt{\frac{\theta\xi}{GE}}(g+e)\right)}{\left(\sqrt{\frac{\theta}{E}}\right)\left(\sqrt{\frac{\theta}{F}}\right)\left(\sqrt{\frac{\theta}{G}}\right)\left(\sqrt{\frac{\theta}{H}}\right)}$$

$$=\frac{\xi.\sqrt{\frac{\xi}{\theta}}.\sqrt{EFGH}.(e+f)(f+g)(g+e)}{EFG}$$

$$=\frac{\left(\frac{\sum_{cyc}e}{r}\right)(e+f)(f+g)(g+e).\sqrt{(e+f)^2(e+g)^2(e+h)^2(f+g)^2(f+h)^2(h+e)^2}}{(e+f)^2(e+g)^2(f+g)^2(h+e)(h+f)(h+g)}$$

$$=\frac{\sum_{cyc}e}{r}\Rightarrow\frac{\sin\frac{A+B}{2}\sin\frac{B+C}{2}\sin\frac{C+A}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2}}+12=\frac{\sum_{cyc}e}{r}+12$$

$$\therefore\left(\frac{\sin\frac{A+B}{2}\sin\frac{B+C}{2}\sin\frac{C+A}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2}}+12\right).\sqrt[4]{r}.\left(\frac{1}{\sqrt[4]{AA'}}+\frac{1}{\sqrt[4]{BB'}}+\frac{1}{\sqrt[4]{CC'}}+\frac{1}{\sqrt[4]{DD'}}\right)$$

$$=\left(\frac{\sum_{cyc}e}{r}+12\right).\sqrt[4]{r}.\sum_{cyc}\frac{1}{\sqrt[4]{e}}\overset{Jensen}{\geq}\left(\frac{\sum_{cyc}e}{r}+12\right).\sqrt[4]{r}.\frac{4}{\sqrt[4]{\frac{\sum_{cyc}e}{4}}}$$

$$\left(\because f(x)=\frac{1}{\sqrt[4]{x}}\text{ is convex as }f''(x)=\frac{5}{16x^{\frac{9}{4}}}>0\right)=\frac{4\left(\frac{\sum_{cyc}e}{r}+12\right)}{\sqrt[4]{\frac{\sum_{cyc}e}{4r}}}=\frac{4(4t^4+12)}{t}$$

$$\left(t=\sqrt[4]{\frac{\sum_{cyc}e}{4r}}\right)\overset{?}{\geq}64\Leftrightarrow t^4-4t+3\overset{?}{\geq}0\Leftrightarrow(t^2+2t+3)(t-1)^2\overset{?}{\geq}0\to\text{true}$$

$$\therefore\left(\frac{\sin\frac{A+B}{2}\sin\frac{B+C}{2}\sin\frac{C+A}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2}}+12\right).\sqrt[4]{r}.\left(\frac{1}{\sqrt[4]{AA'}}+\frac{1}{\sqrt[4]{BB'}}+\frac{1}{\sqrt[4]{CC'}}+\frac{1}{\sqrt[4]{DD'}}\right)\geq 64$$

$$\Rightarrow\sqrt[4]{r}\left(\frac{1}{\sqrt[4]{AA'}}+\frac{1}{\sqrt[4]{BB'}}+\frac{1}{\sqrt[4]{CC'}}+\frac{1}{\sqrt[4]{DD'}}\right)\geq\frac{64}{\frac{\sin\frac{A+B}{2}\sin\frac{B+C}{2}\sin\frac{C+A}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2}}+12}\text{ (QED)}$$