ROMANIAN MATHEMATICAL MAGAZINE

Prove the following inequalities : (i) If $x, y, z > 0$, then :

$$\sum_{cyc} \frac{1}{x+y} \le \sum_{cyc} \frac{4x}{3y^2+2yz+3z^2} \le \frac{\sum_{cyc} x^2}{2xyz}$$ **and (ii) In all triangle ABC**

with usual notations, the following relationship holds :

$$\frac{5s^2+r^2+4Rr}{8s(s^2+2Rr+r^2)} \le \sum_{cyc} \frac{a}{3b^2+2bc+3c^2} \le \frac{s^2-r^2-4Rr}{16sRr}$$

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Assigning $y+z=a, z+x=b, x+y=c \Rightarrow a+b-c=2z>0, b+c-a=2x>0$ and $c+a-b=2y>0 \Rightarrow a+b>c, b+c>a, c+a>b \Rightarrow a,b,c$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say)

yielding $2\sum_{cyc} x = \sum_{cyc} a = 2s \Rightarrow \sum_{cyc} x \overset{(\bullet)}{=} s \Rightarrow x = s-a, y=s-b, z=s-c$

$\therefore xyz \overset{(\bullet\bullet)}{=} r^2 s$ and, $\sum_{cyc} xy = \sum_{cyc}(s-a)(s-b) = 4Rr + r^2 \Rightarrow \sum_{cyc} xy \overset{(\bullet\bullet\bullet)}{=} 4Rr + r^2$

$$\text{Firstly, } \sum_{cyc} \frac{4x}{3y^2+2yz+3z^2} \overset{A-G}{\le} \sum_{cyc} \frac{4x}{8yz} = \frac{\sum_{cyc} x^2}{2xyz}$$

$$\text{Again, } \sum_{cyc} \frac{4x}{3y^2+2yz+3z^2} = \sum_{cyc} \frac{4x^2}{3xy^2+2xyz+3xz^2} \overset{\text{Bergstrom}}{\ge}$$

$$\frac{4\left(\sum_{cyc} x\right)^2}{3\sum_{cyc}\left(xy\left(\sum_{cyc} x - z\right)\right) + 6xyz} = \frac{4\left(\sum_{cyc} x\right)^2}{3\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - 3xyz} \overset{\text{via } (\bullet), (\bullet\bullet) \text{ and } (\bullet\bullet\bullet)}{=}$$

$$\frac{4s^2}{3s(4Rr+r^2)-3r^2 s} \overset{?}{\ge} \sum_{cyc} \frac{1}{x+y} = \sum_{cyc} \frac{1}{a} = \frac{s^2+4Rr+r^2}{4Rrs}$$

$$\Leftrightarrow 4R(s^2-12Rr-3r^2) \overset{?}{\ge} 0 \Leftrightarrow s^2-16Rr+5r^2+4r(R-2r) \overset{?}{\ge} 0 \to \text{true}$$

$$\because s^2-16Rr+5r^2 \overset{\text{Gerretsen}}{\ge} 0 \text{ and } 4r(R-2r) \overset{\text{Euler}}{\ge} 0$$

$$\therefore \sum_{cyc} \frac{4x}{3y^2+2yz+3z^2} \ge \sum_{cyc} \frac{1}{x+y} \text{ and hence,}$$

$$\boxed{\sum_{cyc} \frac{1}{x+y} \le \sum_{cyc} \frac{4x}{3y^2+2yz+3z^2} \le \frac{\sum_{cyc} x^2}{2xyz}} \quad \text{"="} \text{ iff } x=y=z$$

and implementing this with $x \equiv a, y \equiv b, z \equiv c$ ($a, b, c \to$ sides of Δ ABC),

$$\text{we get : } \sum_{cyc} \frac{4a}{3b^2+2bc+3c^2} \le \frac{\sum_{cyc} a^2}{2abc} = \frac{s^2-r^2-4Rr}{4sRr}$$

$$\Rightarrow \sum_{cyc} \frac{a}{3b^2+2bc+3c^2} \le \frac{s^2-r^2-4Rr}{16sRr} \text{ and also, } \sum_{cyc} \frac{4a}{3b^2+2bc+3c^2} \ge \sum_{cyc} \frac{1}{b+c}$$

$$= \frac{1}{2s(s^2+2Rr+r^2)} \cdot \sum_{cyc} \left(a^2 + \sum_{cyc} ab \right)$$

$$= \frac{1}{2s(s^2+2Rr+r^2)} \cdot \left(\left(\sum_{cyc} a^2 + 2\sum_{cyc} ab \right) + \sum_{cyc} ab \right) = \frac{4s^2+s^2+4Rr+r^2}{2s(s^2+2Rr+r^2)}$$

$$= \frac{5s^2+r^2+4Rr}{2s(s^2+2Rr+r^2)} \Rightarrow \sum_{cyc} \frac{a}{3b^2+2bc+3c^2} \ge \frac{5s^2+r^2+4Rr}{8s(s^2+2Rr+r^2)}$$

$$\therefore \boxed{\frac{5s^2+r^2+4Rr}{8s(s^2+2Rr+r^2)} \le \sum_{cyc} \frac{a}{3b^2+2bc+3c^2} \le \frac{s^2-r^2-4Rr}{16sRr}}$$

$'' ='' $ iff Δ ABC is equilateral (QED)