ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{m_a^{k+1}}{m_b+m_c-m_a}\ge\sum_{cyc}m_a^k \text{ for all } k\in\mathbb{N}$$

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$$\text{We shall prove : }\frac{1}{2}\sum_{cyc}\frac{a^{k+1}}{s-a}\ge\sum_{cyc}a^k \text{ for all } k\in\mathbb{N}$$

$$\boxed{\text{Case 1}}\ k=0 \text{ and then : }\frac{1}{6}\sum_{cyc}\frac{a^{k+1}}{s-a}=\frac{1}{6}\sum_{cyc}\frac{a}{s-a}=\frac{1}{6}\sum_{cyc}\frac{a-s+s}{s-a}$$

$$=\frac{1}{6}\left(-3+\frac{s}{r^2s}.\sum_{cyc}(s-b)(s-c)\right)=\frac{1}{6}\left(-3+\frac{(4Rr+r^2)}{r^2}\right)=\frac{2R-r}{3r}\overset{\text{Euler}}{\ge}\frac{4r-r}{3r}$$

$$\Rightarrow\frac{1}{6}\sum_{cyc}\frac{a}{s-a}\overset{(*)}{\ge}1\Rightarrow\frac{1}{2}\sum_{cyc}\frac{a^{k+1}}{s-a}\ge 3=\sum_{cyc}a^k \text{ for } k=0$$

$$\boxed{\text{Case 2}}\ k\in\mathbb{N}^* \text{ and then : }\frac{1}{2}\sum_{cyc}\frac{a^{k+1}}{s-a}\overset{\text{Chebyshev}}{\ge}\frac{1}{6}\left(\sum_{cyc}a^k\right)\left(\sum_{cyc}\frac{a}{s-a}\right)$$

$$\left(\because \text{ WLOG assuming } a\ge b\ge c\Rightarrow a^k\ge b^k\ge c^k \text{ and } \frac{a}{s-a}\ge\frac{b}{s-b}\ge\frac{c}{s-c}\right.$$
$$\left.\text{as } k>0 \text{ since } k\in\mathbb{N}^*\right)$$

$$\overset{\text{via }(*)}{\ge}\sum_{cyc}a^k \text{ for all } k\in\mathbb{N}^*\ \therefore \text{ for all } k\in\mathbb{N},\frac{1}{2}\sum_{cyc}\frac{a^{k+1}}{s-a}\ge\sum_{cyc}a^k$$

$$\Rightarrow\sum_{cyc}\frac{a^{k+1}}{b+c-a}\overset{(**)}{\ge}\sum_{cyc}a^k \text{ and implementing } (**) \text{ on a triangle with sides}$$

$$m_a,m_b,m_c, \text{ we arrive at : }\sum_{cyc}\frac{m_a^{k+1}}{m_b+m_c-m_a}\ge\sum_{cyc}m_a^k$$

for all $k\in\mathbb{N}$ and $\forall\,\Delta$ ABC, " = " iff Δ ABC is equilateral (QED)