ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum_{cyc}\tan\frac{A}{2}\sqrt{\frac{1}{3}\left(\tan^2\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan^2\frac{C}{2}\right)}\geq 1$$

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$Note\ In\ \ any\ \Delta ABC$:

$$\sum\tan\frac{A}{2}\tan\frac{B}{2}=1\ ,Let\tan\frac{A}{2}=x,\tan\frac{B}{2}=y,\tan\frac{C}{2}=z$$

$$Now\sum xy=1\ we\ will\ show:$$

$$(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2)\geq 1$$

$Proof$:

$$(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2)\geq 1$$

$$or\ (x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2)\geq(xy+yz+zx)^3\ \left(as\ \sum xy=1\right)$$

$$(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2)=$$

$$=(xy+x^2+y^2)(y^2+z^2+yz)(x^2+zx+z^2)\overset{Holder}{\geq}$$

$$\geq(xy+yz+zx)^3=1$$

$$WLOG\ a\geq b\geq c\ so\ \tan\frac{A}{2}\geq\ \tan\frac{B}{2}\geq,\tan\frac{C}{2}$$

$$\sum\tan\frac{A}{2}\sqrt{\frac{1}{3}\left(\tan^2\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan^2\frac{C}{2}\right)}\overset{Chebyshev}{\geq}$$

$$\geq\frac{1}{3}\sum\tan\frac{A}{2}\ \sum\sqrt{\frac{1}{3}\left(\tan^2\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan^2\frac{C}{2}\right)}\overset{AM-GM}{\geq}$$

$$\frac{1}{3}\cdot\frac{1}{\sqrt{3}}\frac{4R+r}{s}\ .3\left(\prod\sqrt{\frac{1}{3}\left(\tan^2\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan^2\frac{C}{2}\right)}\right)^{\frac{1}{3}}\overset{\frac{4R+r}{s}\geq\sqrt{3}}{\geq}$$

$$\geq\ \frac{1}{3}\cdot\frac{1}{\sqrt{3}}\ .\sqrt{3}\ .(1)^{\frac{1}{3}}=1$$

Equality holds for $a=b=c$.