ROMANIAN MATHEMATICAL MAGAZINE

Prove that in all triangle ABC with usual notations, the following relationship holds

$$\frac{9}{\sum_{cyc} m_a} \le \sum_{cyc} \frac{2}{m_a + m_b} < \frac{10}{\sum_{cyc} m_a}$$

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$$\sum_{cyc} \frac{2}{a+b} < \frac{10}{\sum_{cyc} a} \Leftrightarrow \frac{1}{(a+b)(b+c)(c+a)} \cdot \sum_{cyc}\left(a^2 + \sum_{cyc} ab\right) < \frac{5}{2s} \Leftrightarrow$$

$$\frac{1}{2s(s^2+2Rr+r^2)}\left(\sum_{cyc} a^2 + 2\sum_{cyc} ab + \sum_{cyc} ab\right) < \frac{5}{2s} \Leftrightarrow \frac{4s^2+s^2+4Rr+r^2}{s^2+2Rr+r^2} < 5$$

$$\Leftrightarrow 10Rr + 5r^2 > 4Rr + r^2 \Leftrightarrow 6Rr + 4r^2 > 0 \to \text{true} \therefore \sum_{cyc} \frac{2}{a+b} < \frac{10}{\sum_{cyc} a}$$

and implementing it on a triangle with sides m_a, m_b, m_c, we arrive at :

$$\sum_{cyc} \frac{2}{m_a + m_b} < \frac{10}{\sum_{cyc} m_a} \text{ and also, } \sum_{cyc} \frac{2}{m_a + m_b} \overset{\text{Bergstrom}}{\ge} \frac{18}{2\sum_{cyc} m_a}$$

$$\therefore \frac{9}{\sum_{cyc} m_a} \le \sum_{cyc} \frac{2}{m_a + m_b} \text{ (QED)}$$