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If in ΔABC holds $\frac{\sum\frac{a}{r_a-r}}{\sin A+\sin B+\sin C}\le 2$ then ΔABC is an equilateral one.

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$$\textit{In an equilateral triangle holds } R=2r$$

$$\textit{Proof}: \textit{Let us assume } \Delta ABC \textit{ is equilateral and}$$

$$AB=BC=CA=a \textit{ and } A=B=C=\frac{\pi}{3}$$

$$a=2R\sin A=2R\sin\frac{\pi}{3}=2R\frac{\sqrt{3}}{2}\ or, R=\frac{a}{\sqrt{3}}\ and\ [ABC]=\sqrt{3}\frac{a^2}{4}, r=\frac{\frac{[ABC]}{a+a+a}}{2}=\frac{a}{2\sqrt{3}}$$

$$\frac{R}{r}=\frac{\frac{a}{\sqrt{3}}}{\frac{a}{2\sqrt{3}}}=2\ or\ R=2r\ \ (1)$$

$$\sum\frac{a}{r_a-r}=\sum\frac{a}{\frac{F}{s-a}-r}=\sum\frac{a}{r.\frac{s}{s-a}-r}=\sum\frac{a(s-a)}{rs-r(s-a)}=\frac{1}{r}\sum\frac{a(s-a)}{a}$$

$$=\frac{1}{r}\sum(s-a)=\frac{1}{r}(3s-2s)=\frac{s}{r}\ \ (2)$$

$$\sin A+\sin B+\sin C=\frac{s}{R}\ \ (3)$$

$$\frac{\sum\frac{a}{r_a-r}}{\sin A+\sin B+\sin C}\le 2\ or,\frac{\left(\frac{s}{r}\right)}{\frac{s}{R}}\le 2\ \big(using\ \ (3)\&\ \ (2)\big)or$$

$$\frac{R}{r}\le 2\ ,we\ \ know\ that\frac{R}{r}\ge 2\ (Euler)$$

$$So\ \ we\ can\ \ say\ \frac{R}{r}=2\ ,and\ \ using\ (1)\ we\ \ can\ \ say\ \Delta ABC\ \ equilateral$$