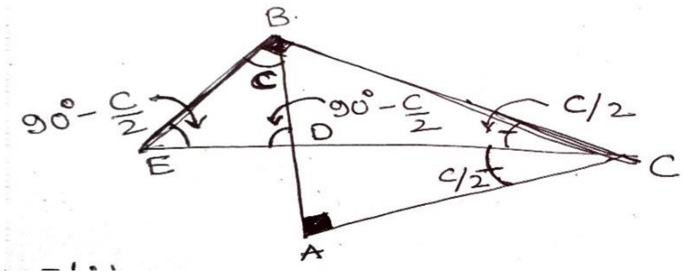
ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC $\angle A = 90°$, CD − internal bisector,

$D \in (AB)$, BE perpendicular on BC,

$CD \cap BE = \{E\}$, R_1, R_2 − Circumradii of ΔBCE, ΔBDE.

Prove that:



$$2[BDE] < BD^2 < \frac{(R_1+R_2)^2}{2}$$

Proposed by Radu Diaconu-Romania

Solution by Tapas Das-India

$$Clearly, \angle EBC = \angle DAC = 90°, \angle BCE = \angle DCA = \frac{C}{2} \text{ and}$$

$$\angle BED = \angle ADC = \angle BDE = 90° - \frac{C}{2}, \text{ from } \Delta EBD \text{ we have:}$$

$$BE = BD(\text{Since } \angle BED = \angle BDE)\ (1) \text{ and } \angle EBD = C$$

$$[BDE] = \frac{1}{2} BE.BD.\sin\angle EBD \overset{(1)}{=} \frac{1}{2} BD^2 \sin C \text{ or}$$

$$2[BDE] = BD^2 \sin C < BD^2 (as\ \sin C \le 1)\ (*)$$

$$\text{from } \Delta BDE, 2R_2 = \frac{BD}{\sin\angle BED} = \frac{BD}{\sin\left(90° - \frac{C}{2}\right)} = \frac{BD}{\cos\frac{C}{2}} \text{ or, } 2R_2 \cos\frac{C}{2} = BD\ (2)$$

$$\text{from } \Delta BEC, 2R_1 = \frac{BE}{\sin\frac{C}{2}} \overset{(1)}{=} \frac{BD}{\sin\frac{C}{2}} \text{ or, } 2R_1 \sin\frac{C}{2} = BD\ (3)$$

$$Now\ from\ (2)\&\ (3)\ we\ get\ BD^2 = BD.BD = \left(2R_2\cos\frac{C}{2}\right)\left(2R_1\sin\frac{C}{2}\right)$$

$$= 2R_1R_2\left(2\sin\frac{C}{2}\cos\frac{C}{2}\right) = 2R_1R_2\sin C$$

$$< 2R_1R_2(\ as\ \sin C < 1) \overset{AM-GM}{<} 2\left(\frac{R_1+R_2}{2}\right)^2 = \frac{(R_1+R_2)^2}{2} (**)$$

$$Now\ from\ (*)\&(**) we\ have\ 2[BDE] < BD^2 < \frac{(R_1+R_2)^2}{2}$$