ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\pi\left(\frac{3}{2}+\frac{r}{R}\right)<\left(\sqrt{A.\frac{h_b+h_c}{h_a}}+\sqrt{B.\frac{h_c+h_a}{h_b}}+\sqrt{C.\frac{h_a+h_b}{h_c}}\right)^2\le\frac{\pi}{4r^2}(5R^2-Rr+6r^2)$$

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$$We\ know\ that\ \forall x,y,z>0,(x+y+z)^2>x^2+y^2+z^2$$

$$WLOG\ a\ge b\ge c\ \ then\ A\ge B\ge C\ and\ \frac{1}{c}\ge\frac{1}{b}\ge\frac{1}{a}$$

$$\left(\sqrt{A.\frac{h_b+h_c}{h_a}}+\sqrt{B.\frac{h_c+h_a}{h_b}}+\sqrt{C.\frac{h_a+h_b}{h_c}}\right)^2>$$

$$>A\cdot\frac{h_b+h_c}{h_a}+B\cdot\frac{h_c+h_a}{h_b}+C\cdot\frac{h_a+h_b}{h_c}=$$

$$=A\left(\frac{a}{b}+\frac{a}{c}\right)+B\left(\frac{b}{c}+\frac{b}{a}\right)+C\left(\frac{c}{a}+\frac{c}{b}\right)\overset{Chebyshev}{\ge}$$

$$\ge\frac{1}{3}(A+B+C)\left(\left(\frac{a}{b}+\frac{a}{c}\right)+\left(\frac{b}{c}+\frac{b}{a}\right)+\left(\frac{c}{a}+\frac{c}{b}\right)\right)\overset{AM-GM}{>}$$

$$>\frac{1}{3}\pi 6\sqrt{\left(\frac{a}{b}.\frac{a}{c}\right).\left(\frac{b}{c}.\frac{b}{a}\right).\left(\frac{c}{a}.\frac{c}{b}\right)}=2\pi=\pi\left(\frac{3}{2}+\frac{1}{2}\right)\overset{Euler}{>}\pi\left(\frac{3}{2}+\frac{r}{R}\right)$$

$$\left(\sqrt{A.\frac{h_b+h_c}{h_a}}+\sqrt{B.\frac{h_c+h_a}{h_b}}+\sqrt{C.\frac{h_a+h_b}{h_c}}\right)^2\overset{C-S}{\le}(A+B+C)\sum\frac{h_b+h_c}{h_a}=$$

$$=\pi\sum\left(\frac{a}{b}+\frac{a}{c}\right)=\pi\sum\left(\frac{a}{b}+\frac{b}{a}\right)\overset{Bandila}{\le}\frac{3\pi R}{r}$$

$$We\ \ need\ to\ show\ \frac{3\pi R}{r}\le\frac{\pi}{4r^2}(5R^2-Rr+6r^2)\ or$$

$$5R^2-13Rr+6r^2\ge 0\ or\ (R-2r)(5R-3r)\ge 0\ True\ (Euler)$$

Equality holds for $a=b=c$.