ROMANIAN MATHEMATICAL MAGAZINE

If in $\Delta ABC, R=7r$ then:

$$l_a^2+l_b^2+l_c^2\geq \frac{9}{14}\frac{(abc)^{\frac{4}{3}}}{\max(a^2,b^2,c^2)}$$

Proposed by Radu Diaconu-Romania

Solution by Tapas Das-India

$$l_a^2+l_b^2+l_c^2\geq h_a^2+h_b^2+h_c^2=\frac{b^2c^2+a^2c^2+a^2b^2}{4R^2}\overset{AM-GM}{\geq}\frac{3(abc)^{\frac{4}{3}}}{4R^2}=$$

$$\overset{R=7r}{=}\frac{3(abc)^{\frac{4}{3}}}{4R.7r}=\frac{3}{14}\frac{(abc)^{\frac{4}{3}}}{2Rr}$$

$$We\ need\ to\ show:$$

$$\frac{3}{14}\frac{(abc)^{\frac{4}{3}}}{2Rr}\geq\frac{9}{14}\frac{(abc)^{\frac{4}{3}}}{\max(a^2,b^2,c^2)},\quad \max(a^2,b^2,c^2)\geq 6Rr$$

$$\frac{a^2+b^2+c^2}{3}\geq 6Rr,\ \ 2(s^2-r^2-4Rr)\geq 18Rr$$

$$s^2\geq 13Rr+r^2,\quad 16Rr-5r^2\geq 13Rr+r^2$$

$$3Rr\geq 6r^2,\quad R\geq 2r\ Euler$$

Equality holds for: $a=b=c$.