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In ΔABC the following relationship holds:

$$A\cdot\frac{a^2}{bc}+B\cdot\frac{b^2}{ac}+C\cdot\frac{c^2}{ab}\geq 4\pi\ \min\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right)$$

Proposed by Radu Diaconu-Romania

Solution by Tapas Das-India

$$A\cdot\frac{a^2}{bc}+B\cdot\frac{b^2}{ac}+C\cdot\frac{c^2}{ab}=A\cdot\frac{a^3}{abc}+B\cdot\frac{b^3}{abc}+C\cdot\frac{c^3}{abc}=$$

$$=\frac{1}{abc}(A.a^3+B.b^3+C.c^3)\overset{Chebyshev}{\geq}\frac{1}{abc}\cdot\frac{1}{3}(A+B+C)(a^3+b^3+c^3)=$$

$$=\frac{1}{12Rrs}\pi\cdot 2s(s^2-3r^2-6Rr)\overset{Gerretsen}{\geq}\frac{\pi}{6Rr}(16Rr-5r^2-3r^2-6Rr)=$$

$$=\frac{\pi}{6Rr}(10Rr-8r^2)$$

We need to show:

$$\frac{\pi}{6Rr}(10Rr-8r^2)\geq 4\pi\min\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right)$$

$$\frac{\pi}{6Rr}(10Rr-8r^2)\geq 4\pi\cdot\frac{1}{3}\sum\sin^2\frac{A}{2}$$

$$or\ \frac{10Rr-8r^2}{6Rr}\geq\frac{4}{3}\frac{(2R-r)}{2R}\ or\ 10Rr-8r^2\geq 8Rr-4r^2$$

$$2Rr\geq 4r^2\ or\ R\geq 2r\ (Euler)$$

Equality holds for an equilateral triangle.