ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c\geq 0, a+b+c=2.$ Find the minimum value of

$$P=\sqrt{a^2+b^2+7c}+\sqrt{b^2+c^2+7a}+\sqrt{c^2+a^2+7b}.$$

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We will prove that the minimum of P is $2\sqrt{5}$, achieved at

$a=b=\frac{2}{3}$ and when one of a,b,c is 0 and the others are 1. Let $p:=a+b+c=2$,

$q:=ab+bc+ca,\quad r:=abc.$ By Hölder's inequality, we have

$$\left(\sum_{cyc}\sqrt{b^2+c^2+7a}\right)^2.\sum_{cyc}\frac{(a+1)^3}{b^2+c^2+7a}\geq\left(\sum_{cyc}(a+1)\right)^3=125,$$

$$\sum_{cyc}\frac{(a+1)^3}{b^2+c^2+7a}=\frac{544-221q+q^2-4q^3+(768-123q+10q^2)r+32r^2}{224-84q-10q^2-2q^3+(313-27q)r-r^2}.$$

$$P^2\geq 125.\frac{224-84q-10q^2-2q^3+(313-27q)r-r^2}{544-221q+q^2-4q^3+(768-123q+10q^2)r+32r^2}\overset{?}{\geq}\left(2\sqrt{5}\right)^2$$

$$\Leftrightarrow\ f(r)=32+22q-52q^2-2q^3+(29+111q-20q^2)r-69r^2\geq 0.$$

We have $q\leq\frac{p^2}{3}$ and $pq\geq 9r$, then $q\leq\frac{4}{3},\quad r\leq\frac{2q}{9}$, and

$$f'(r)=29+111q-20q^2-138r>29+111q-20q^2-138.\frac{2q}{9}>0,$$

so f is increasing, and if $q\leq 1$, we have

$$f(r)\geq f(0)=32+22q-52q^2-2q^3=(1-q)(32+54q+2q^2)\geq 0.$$

If $1\leq q\leq\frac{4}{3}$, we have by Schur's inequality, $r\geq\frac{p(4q-p^2)}{9}=\frac{8(q-1)}{9}$, and

$$f(r)\geq f\left(\frac{8(q-1)}{9}\right)=\frac{2}{9}(q-1)\left(\frac{4}{3}-q\right)(163+89q)\geq 0.$$

which completes the proof.