ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c\geq 0, ab+bc+ca=1.$ Maximize the following expression :

$$P=\frac{1}{a+2ab+3}+\frac{1}{b+2bc+3}+\frac{1}{c+2ca+3}$$

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We will prove that the maximum value of P is $\frac{16}{21}$. We have :

$$P\leq\frac{16}{21}\Leftrightarrow\sum_{cyc}\frac{1}{a+2ab+3}\leq\frac{16}{21}\Leftrightarrow 21\sum_{cyc}(b+2bc+3)(c+2ca+3)$$
$$\leq 16\prod_{cyc}(a+2ab+3)$$

$$\Leftrightarrow a+b+c+3\left(ab^2+bc^2+ca^2\right)+\frac{121}{9}abc+\frac{70}{9}abc(a+b+c)+\frac{64}{9}(abc)^2\geq 4.\ (1)$$

By AM − GM inequality, we have

$$16ab^2+a(2-c)^2\geq 8ab(2-c).$$

Adding this inequality with its similar ones, we obtain

$$16\left(ab^2+bc^2+ca^2\right)+4(a+b+c)+\left(ac^2+ba^2+cb^2\right)+24abc\geq 20$$

$$\Leftrightarrow 15\left(ab^2+bc^2+ca^2\right)+(a+b+c)(ab+bc+ca)+4(a+b+c)+21abc\geq 20$$

$$\Leftrightarrow a+b+c+3\left(ab^2+bc^2+ca^2\right)+\frac{21}{5}abc\geq 4.$$

So the inequality (1) is true and the proof is complete.

The maximum value of P is $\frac{16}{21}$ achieved at $(a,b,c)\in\left\{\left(2,\frac{1}{2},0\right),\left(0,2,\frac{1}{2}\right),\left(\frac{1}{2},0,2\right)\right\}$.