PP37343

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If $x, y, z \in \mathbb{R}_+^* = (0, \infty)$ then in any ΔABC with the area F the following inequality holds:

$$\frac{y+z}{x \cdot h_b^2} c^2 + \frac{z+x}{y \cdot h_c^2} a^2 + \frac{x+y}{z \cdot h_c^2} b^2 \geq 8$$

Solution by Rovsen Pirguliyev - Azerbaijan.

$$F = \frac{1}{2} a h_a = \frac{1}{2} b h_b = \frac{1}{2} c h_c, \text{ then we have:}$$

$$\frac{y+z}{x} \cdot \left(\frac{bc}{bh_b}\right)^2 + \frac{z+x}{y} \cdot \left(\frac{ac}{ch_c}\right)^2 + \frac{x+y}{z} \cdot \left(\frac{ab}{ah_a}\right)^2 =$$

$$= \frac{1}{4F^2} \cdot \left[\left(\frac{y+z}{x} + 1 - 1\right) \cdot (bc)^2 + \left(\frac{z+x}{y} + 1 - 1\right) \cdot (ac)^2 + \left(\frac{x+y}{z} + 1 - 1\right)(ab)^2\right] =$$

$$= \frac{1}{4F^2} \cdot \left[\frac{y+z+x}{x}(bc)^2 + \frac{z+x+y}{y} \cdot (ac)^2 + \frac{x+y+z}{z}(ab)^2 - ((bc)^2 + (ac)^2 + (ab)^2)\right] \geq$$

$$\geq \frac{1}{4F^2}\left[(x+y+z) \cdot \frac{(bc+ac+ab)^2}{x+y+z} - ((bc)^2 + (ac)^2 + (ab)^2)\right] =$$

$$= \frac{1}{4F^2} \cdot (2ab^2c + 2abc^2 + 2a^2bc) = \frac{1}{2F^2} \cdot abc(a+b+c) \overset{\text{AM-GM}}{\geq}$$

$$\geq \frac{1}{2F^2} \cdot abc \cdot 3\sqrt[3]{abc} = \frac{3}{2F^2} \sqrt[3]{(abc)^4}$$

(using the inequality $(abc)^2 \geq \left(\frac{4F}{\sqrt{3}}\right)^3$, Geometric inequalities 4.14 Bottema)

$$\Rightarrow LHS \geq \frac{3}{2F^2} \sqrt[3]{(abc)^4} \geq \frac{3}{2F^2} \cdot \sqrt[3]{\left(\frac{4F}{\sqrt{3}}\right)^6} = \frac{3}{2F^2} \cdot \left(\frac{4F}{\sqrt{3}}\right)^2 = 8$$

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