PP39414

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If $x, y, z \in \mathbb{R}_+^* = (0, \infty)$, then in any ΔABC triangle with the area F, the following inequality holds:

$$\frac{y+z}{x} \cdot \frac{b^7+c^7}{b^5+c^5} + \frac{z+x}{y} \cdot \frac{c^7+a^7}{c^5+a^5} + \frac{x+y}{z} \cdot \frac{a^7+b^7}{a^5+b^5} \geq 8\sqrt{3}F$$

Solution by Rovsen Pirguliyev - Azerbaijan.
To prove that:

$$\frac{b^7+c^7}{b^5+c^5} \geq \frac{b^2+c^2}{2} \quad (1)$$

$$(1) \Leftrightarrow 2b^7 + 2c^7 \geq b^7 + c^7 + c^2b^5 + c^5b^2 \Leftrightarrow (b^5 - c^5)(b^2 - c^2) \geq 0$$

it is true, since $b^5 - c^5$ and $b^2 - c^2$ has the same sign. Or

$$(b-c)(b^4 + \ldots + c^4)(b-c)(b+c) \geq 0$$

or

$$(b-c)^2(b+c)(b^4 + \ldots + c^4) \geq 0$$

Using (1), AM-GM ($x + y + z \geq 3\sqrt[3]{xyz}$) and $(abc)^2 \geq (\frac{4F}{\sqrt{3}})^3$ (L. Carlitz 4.14. Geometric inequality, Bottema), we have:

$$LHS \geq \frac{1}{2} \cdot \frac{2\sqrt{yz}}{x} \cdot 2bc + \frac{1}{2} \cdot \frac{2\sqrt{zx}}{y} \cdot 2ca + \frac{1}{2} \cdot \frac{2\sqrt{xy}}{z} \cdot 2ab \geq$$

$$\geq 3\sqrt[3]{8(abc)^2} \geq 3 \cdot 2 \cdot \sqrt[3]{\left(\frac{4F}{\sqrt{3}}\right)} = 6 \cdot \frac{4F}{\sqrt{3}} = 8\sqrt{3}F$$

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