PP40009

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If $t \in (0, \sqrt{3})$, then:

$$\left(\arctan^2 t + \arctan^2\left(\frac{\sqrt{3}-t}{1+\sqrt{3}t}\right) + \frac{\pi^2}{9}\right)^2 = 2\left(\arctan^4 t + \arctan^4\left(\frac{\sqrt{3}-t}{1+\sqrt{3}t}\right) + \frac{\pi^4}{81}\right)$$

Solution by Rovsen Pirguliyev - Azerbaijan.
To prove that

$$(1) \qquad (a^2+b^2+(a+b)^2)^2 = 2(a^4+b^4+(a+b)^4) \qquad \text{(Candido), Brute force!}$$

$$\begin{aligned}(a^2+b^2+(a+b)^2)^2 &= (a^2+b^2)^2 + 2(a^2+b^2)(a+b)^2 + (a+b)^4 = \\ &= (a^2+b^2)^2 + 2(a^2+b^2)(a^2+b^2+2ab) + (a+b)^4 = \\ &= (a^2+b^2)^2 + 2(a^2+b^2)^2 + 4ab(a^2+b^2) + (a+b)^4 = \\ &= (a+b)^4 + 3(a^2+b^2)^2 + 4ab(a^2+b^2) = \\ &= (a+b)^4 + \underbrace{a^4+4a^3b+4ab^3+6a^2b^2+b^4}_{(a+b)^4} + 2a^4 + 2b^4 = \\ &= 2(a+b)^4 + 2(a^4+b^4) = 2(a^4+b^4+(a+b)^4))\end{aligned}$$

$$\text{take } a \to \arctan t \qquad b \to \arctan\frac{\sqrt{3}-t}{1+\sqrt{3}t}$$

$$a+b = \frac{\pi}{3} \text{ (prove: } \arctan t + \arctan\frac{\sqrt{3}-t}{1+\sqrt{3}t} = \frac{\pi}{3}; \text{ since } t \in (0;\sqrt{3}) \Rightarrow$$

$$\Rightarrow \tan\left(\arctan t + \arctan\frac{\sqrt{3}-t}{1+\sqrt{3}t}\right) = \tan\frac{\pi}{3}$$

$$\Rightarrow \frac{t + \frac{\sqrt{3}-t}{1+\sqrt{3}t}}{1 - \frac{t(\sqrt{3}-t)}{1+\sqrt{3}t}} = \sqrt{3} \Rightarrow \frac{\sqrt{3}(t^2+1)}{1+t^2} = \sqrt{3}, \text{ true!)}$$

Then, using (1) we have:

$$\left(\arctan^2 t + \arctan^2\left(\frac{\sqrt{3}-t}{1+\sqrt{3}t}\right) + \frac{\pi^2}{9}\right)^2 = 2\left(\arctan^4 t + \arctan^4\left(\frac{\sqrt{3}-t}{1+\sqrt{3}t}\right) + \frac{\pi^4}{81}\right)$$

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