PP40424

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If $f:\mathbb{R}\to\mathbb{R}, 1+f(x+y)\le f(x)+f(y)\le x+y+2, \forall x,y\in\mathbb{R}$, then in ΔABC holds:

$$\frac{f(x^2)}{y+z}\cdot a^2+\frac{f(y^2)}{z+x}\cdot b^2+\frac{f(z^2)}{x+y}\cdot c^2\ge 4\sqrt{3}F;\quad \forall x,y,z\in(0,\infty)$$

Solution by Rovsen Pirguliyev - Azerbaijan.

In the problem PP40423 we have already proven that $f(x)=x+1$. Then we have:

$$(1)\quad \frac{x^2+1}{y+z}\cdot a^2+\frac{y^2+1}{z+x}\cdot b^2+\frac{z^2+1}{x+y}\cdot c^2\overset{\text{AM-GM}}{\ge}\frac{2x}{y+z}\cdot a^2+\frac{2y}{z+x}\cdot b^2+\frac{2z}{x+y}\cdot c^2$$

Take a triangle $A_1B_1C_1$ with sides $u=\sqrt{a}, v=\sqrt{b}, w=\sqrt{c}$ (area F_1), here a,b,c are the sides of the triangle ABC. It is known that $F_1=\frac{1}{2}\sqrt{(r_a+r_b+r_c)r}$. To prove that

$$(2)\qquad F_1\ge\frac{\sqrt[4]{3F^2}}{2}$$

$$F_1=\frac{1}{2}\sqrt{\underbrace{(r_a+r_b+r_c)}_{4R+r}r}=\frac{1}{2}\sqrt{(4R+r)r}\ge\frac{1}{2}\sqrt{\sqrt{3}rs}=\frac{\sqrt[4]{3F^2}}{2}$$

$$(1)\Rightarrow 2\Big(\frac{xu^4}{y+z}+\frac{yv^4}{z+x}+\frac{zw^4}{x+y}\Big)=2\Big(\frac{(xu^2)^2}{xy+xz}+\frac{(yv^2)^2}{yz+xy}+\frac{(zw^2)^2}{xz+yz}\Big)\overset{\text{Bergström}}{\ge}$$

$$(3)\qquad \ge 2\cdot\frac{(xu^2+yv^2+zw^2)^2}{2(xy+yz+zx)}$$

□

Applying Oppenheim's inequality in (3)

$(xa^2+yb^2+zc^2)\ge 4F\cdot\sqrt{xy+yz+zx}$ for $a\to u, b\to v, c\to w)$

We have: $(3)\Rightarrow 2\cdot\dfrac{(xu^2+yv^2+zw^2)^2}{2(xy+yz+zx)}\ge\dfrac{16F_1^2(xy+yz+zx)}{xy+yz+zx}=$

$$=16F_1^2\overset{(2)}{\ge}16\cdot\Big(\frac{\sqrt[4]{3F^2}}{2}\Big)=8\cdot 4\sqrt{3}F$$

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