PP40441

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If $x, y, z \geq 0$ then in ΔABC the following relationship holds:

$$\frac{a^2e^{x^2}}{y+z} + \frac{b^2e^{y^2}}{z+x} + \frac{c^2e^{z^2}}{x+y} > 4\sqrt{3} \cdot F$$

Solution by Rovsen Pirguliyev - Azerbaijan.

Using $e^x \geq x+1 \quad (x \geq 0)$ we have:

(1)

$$LHS \geq \frac{a^2(x^2+1)}{y+z} + \frac{b^2(y^2+1)}{x+z} + \frac{c^2(z^2+1)}{x+y} \overset{\text{AM-GM}}{\geq} \frac{2x}{y+z}a^2 + \frac{2y}{z+x}b^2 + \frac{2z}{x+y}c^2$$

Take a triangle $A_1B_1C_1$ with sides $u = \sqrt{a}, v = \sqrt{b}, w = \sqrt{c}$ in area F_1, a, b, c are the sides of the ΔABC.

Lemma.

$$F_1 = \frac{1}{2}\sqrt{(r_a + r_b + r_c)r} \qquad \text{(Mehmet Şahin)}$$

To prove that

$$F_1 \geq \frac{\sqrt[4]{3F^2}}{2} \tag{2}$$

$$F_1 = \frac{1}{2}\sqrt{(r_a + r_b + r_c)r} = \frac{1}{2}\sqrt{(4R+r)r} \geq \frac{1}{2}\sqrt{\sqrt{3}sr} = \frac{\sqrt[4]{3F^2}}{2}$$

$$\text{Hence } 2\Big(\frac{x^2a^2}{(y+z)x} + \frac{y^2b^2}{(z+x)y} + \frac{z^2c^2}{z(x+y)}\Big) \overset{\text{in } \Delta A_1B_1C_1}{=}$$

$$= 2 \cdot \Big(\frac{(xu^2)^2}{xy+xz} + \frac{(yv^2)^2}{yz+xy} + \frac{(zw^2)^2}{xz+yz}\Big) \geq 2 \cdot \frac{(xu^2+yv^2+zw^2)^2}{2(xy+yz+zx)} \tag{*}$$

Using Oppenheim's inequality in (*):

$$(xa^2 + yb^2 + zc^2) \geq 4F\sqrt{xy+yz+zx} \text{ for } a \to u, b \to v, c \to w)$$

$$\Rightarrow (*) \Rightarrow 2 \cdot \frac{(xu^2+yv^2+zw^2)^2}{2(xy+yz+zx)} \geq \frac{16F_1^2(xy+yz+zx)}{xy+yz+zx} =$$

$$= 16F_1^2 \geq 16\Big(\frac{\sqrt[4]{3F^2}}{2}\Big)^2 = 4\sqrt{3}F$$

□

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