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D. M. BĂTINEŢU - GIURGIU, DANIEL SITARU - ROMANIA

If $x, y, z > 0$ and $xyz = 1$ then:

$$\left(\left(x+y-\frac{1}{\sqrt{z}}\right)^4+2\right)\left(\left(y+z-\frac{1}{\sqrt{x}}\right)^4+2\right)\left(\left(z+x-\frac{1}{\sqrt{y}}\right)^4+2\right)\geq 27$$

Solution by Rovsen Pirguliyev - Azerbaijan.
Using AM-GM and $xyz = 1$, we have:

$$x+y-\frac{1}{\sqrt{z}} \geq 2\sqrt{xy}-\frac{1}{\sqrt{z}} = \frac{1}{\sqrt{z}}$$

and similarly we have

$$y+z-\frac{1}{\sqrt{x}} \geq \frac{1}{\sqrt{x}}, \quad z+x-\frac{1}{\sqrt{y}} \geq \frac{1}{\sqrt{y}}$$

Using the inequality:

$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca)$$

(APMO-2004), then we have:

$$(1) \qquad LHS \geq \left(\frac{1}{z^2}+2\right)\left(\frac{1}{x^2}+2\right)\left(\frac{1}{y^2}+2\right) \geq 9\left(\frac{1}{xz}+\frac{1}{xy}+\frac{1}{zy}\right)$$

now using AM-GM inequality in (1):

$$a+b+c \geq 3\sqrt[3]{abc}$$

we have:

$$LHS \geq 9\left(\frac{1}{xz}+\frac{1}{xy}+\frac{1}{zy}\right) \geq 9 \cdot 3\sqrt[3]{\frac{1}{x^2y^2z^2}} \overset{xyz=1}{=} 27$$

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Mathematics Department, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania
Email address: dansitaru63@yahoo.com