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D.M. BĂTINEŢU - GIURGIU, DANIEL SITARU - ROMANIA

If $m \geq 0$ then in ΔABC holds:

$$\frac{a^m b}{h_a \cdot h_b^m} + \frac{b^m c}{h_b \cdot h_c^m} + \frac{c^m a}{h_c \cdot h_a^m} \geq 2^{m+1}(\sqrt{3})^{1-m}$$

Solution by Rovsen Pirguliyev - Azerbaijan.
Applying AM-GM inequality, we have:

$$LHS \geq 3\sqrt[3]{\frac{(abc)^{m+1}}{(h_a h_b h_c)^{m+1}}} = 3\sqrt[3]{\frac{((abc)^2)^{\frac{m+1}{2}}}{((h_a h_b h_c)^{\frac{1}{3}})^{3(m+1)}}} \tag{1}$$

further, we apply the inequality

$$(h_a h_b h_c)^{\frac{1}{3}} \leq 3^{\frac{1}{4}} F^{\frac{1}{2}} \tag{6.27}$$

and

$$(abc)^2 \geq \left(\frac{4F}{\sqrt{3}}\right)^3 \tag{4.14}$$

(Geometric inequalities, Bottema (1968)), we have:

$$3\sqrt[3]{\frac{((abc)^2)^{\frac{m+1}{2}}}{((h_a h_b h_c)^{\frac{1}{3}})^{3(m+1)}}} \geq 3\sqrt[3]{\frac{\left(\left(\frac{4F}{\sqrt{3}}\right)^3\right)^{\frac{m+1}{2}}}{\left(3^{\frac{1}{4}} F^{\frac{1}{2}}\right)^{3(m+1)}}} = 3 \cdot \frac{2^{m+1} \cdot F^{\frac{m+1}{2}} \cdot 3^{-\frac{m+1}{4}}}{3^{\frac{m+1}{4}} \cdot F^{\frac{m+1}{2}}} =$$

$$= 2^{m+1} \cdot 3^{\frac{1-m}{2}} = 2^{m+1} \cdot (\sqrt{3})^{1-m}$$

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Mathematics Department, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania
Email address: dansitaru63@yahoo.com