PP40568

D.M. BĂTINEŢU - GIURGIU, DANIEL SITARU - ROMANIA

In ΔABC the following relationship holds:

$$(4R+r-\sqrt{3}a)^2+(4R+r-\sqrt{3}b)^2+(4R+r-\sqrt{3}c)^2 \geq 27r^2$$

Solution by Rovsen Pirguliyev - Azerbaijan.
It is known that

$$\sum r_a = 4R+r \tag{1}$$

and

$$(a+b+c)\sqrt{3} \leq 2(r_a+r_b+r_c) \qquad (S.29) \tag{2}$$

(Geometric inequalities, Bottema, 1968)

$$(1);(2) \Rightarrow 4R+r = r_a+r_b+r_c \geq \frac{(a+b+c)\sqrt{3}}{2} = s\sqrt{3}$$

i.e.

$$4R+r \geq s\sqrt{3} \tag{3}$$

Using (3), we have:

$$LHS \geq (s\sqrt{3}-a\sqrt{3})^2+(s\sqrt{3}-\sqrt{3}b)^2+(\sqrt{3}s-\sqrt{3}c)^2 =$$

$$3\cdot\left[(s-a)^2+(s-b)^2+(s-c)^2\right] \overset{\text{AM-GM}}{\geq} 3\cdot 3\sqrt[3]{(s-a)^2(s-b)^2(s-c)^2} \tag{4}$$

We have: $F=\sqrt{s(s-a)(s-b)(s-c)}$ (Heron)

$$\frac{F^2}{s}=(s-a)(s-b)(s-c) \Rightarrow \frac{F^4}{s^2}=(s-a)^2(s-b)^2(s-c)^2 \Rightarrow$$

$$\Rightarrow (s-a)^2(s-b)^2(s-c)^2 = \frac{s^4r^4}{s^2} = s^2r^4 \geq 27r^6 \tag{5}$$

$$\text{therefore } (4) \Rightarrow LHS \geq 9\sqrt[3]{(s-a)^2(s-b)^2(s-c)^2} \overset{(5)}{\geq}$$
$$\geq 9\sqrt[3]{27r^6} = 27r^2$$

□

Mathematics Department, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania
Email address: dansitaru63@yahoo.com