PP40569

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If $x, y, z > 0$ then in ΔABC holds:

$$\frac{x+y}{z}(s-a)^4 + \frac{y+z}{x}(s-b)^4 + \frac{z+x}{y}(s-c)^4 \geq 54r^4$$

Solution by Rovsen Pirguliyev - Azerbaijan.
Twice applying AM-GM inequality, we have:

$$LHS \geq 3\sqrt[3]{\frac{x+y}{z} \cdot \frac{y+z}{x} \cdot \frac{z+x}{y} \cdot (s-a)^4 \cdot (s-b)^4 \cdot (s-c)^4} \geq$$

$$\geq \sqrt[3]{\frac{2\sqrt{xy}}{z} \cdot \frac{2\sqrt{yz}}{x} \cdot \frac{2\sqrt{zx}}{y} \cdot (s-a)^4 \cdot (s-b)^4 \cdot (s-c)^4} =$$

$$= 6\sqrt[3]{(s-a)^4 \cdot (s-b)^4 \cdot (s-c)^4} \tag{1}$$

We have: $F = \sqrt{s(s-a)(s-b)(s-c)}$ (Heron)

$$\frac{F^2}{s} = (s-a)(s-b)(s-c) \Rightarrow \frac{F^8}{s^4} = (s-a)^4(s-b)^4(s-c)^4 \Rightarrow$$

$$\Rightarrow (s-a)^4(s-b)^4(s-c)^4 = \frac{s^8r^8}{s^4} = s^4r^8 \geq 3^6 \cdot r^{12} \tag{2}$$

$$\text{therefore } LHS \geq 6\sqrt[3]{(s-a)^4(s-b)^4(s-c)^4} \overset{(2)}{\geq} 6\sqrt[3]{3^6r^{12}} = 54r^4$$

$\square$

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