PP40633

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If $x, y \geq 0, a, b, c, x+y > 0$ then holds:

$$((ax+by)^2+2)((bx+cy)^2+2)((cx+ay)^2+2) \geq 9(x+y)^2(ab+bc+ca)$$

Solution by Rovsen Pirguliyev - Azerbaijan.
To prove that

$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca) \tag{1}$$

(APMO-2004). Brute force!

$$LHS = a^2b^2c^2 + 2a^2b^2 + 2b^2c^2 + 2c^2a^2 + 4a^2 + 4b^2 + 4c^2 + 8$$

then we have:

$$a^2b^2c^2 + 2a^2b^2 + 2b^2c^2 + 2c^2a^2 + 4a^2 + 4b^2 + 4c^2 + 8 \geq 9(ab+bc+ca) \Leftrightarrow$$
$$\Leftrightarrow 2a^2b^2 + 2b^2c^2 + 2c^2a^2 + 6 \geq 4a + 4bc + 4ac$$

further it is obvious from by Schur and AM-GM.
Take in (1) $\quad a \to ax+by, b \to bx+cy, c \to cx+ay$, we have:

$$LHS \geq 9((ax+by)(bx+cy) + (bx+cy)(cx+ay) + (ax+by)(cx+ay)) =$$
$$= 9(abx^2+acxy+b^2xy+bcy^2+bcx^2+abxy+c^2xy+acy^2+acx^2+a^2xy+bcxy+aby^2) =$$
$$= 9(x^2(ab+bc+ac) + xy(ac+b^2+ab+c^2+a^2+bc) + y^2(bc+ac+ab)) \tag{2}$$

It is known that:

$$a^2+b^2+c^2 \geq ab+ac+bc \tag{3}$$

$$(a^2+b^2+c^2 \geq ab+ac+bc \Leftrightarrow \frac{1}{2}(a-b)^2 + \frac{1}{2}(b-c)^2 + \frac{1}{2}(a-c)^2 \geq 0)$$

Using (3) in (2) $\Rightarrow$

$$\Rightarrow LHS \geq 9(x^2(ab+bc+ac) + 2xy(ab+bc+ca) + y^2(ab+bc+ca)) =$$
$$= 9(x^2+2xy+y^2)(ab+bc+ac) = 9(x+y)^2 \cdot (ab+bc+ca)$$

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