PP43109

MIHÁLY BENCZE - ROMANIA

If $a, b > 0$ then:

$$(2a^2+b^2)(2a+b)+(a^2+2b^2)(a+2b) \geq 3\sqrt{ab}(a+b)(a+\sqrt{ab}+b)$$

Solution by Daniel Sitaru, Claudia Nănuţi.

$$(2a^2+b^2)(2a+b) = (a^2+a^2+b^2)(a+b+a) \overset{\text{CBS}}{\geq}$$
$$\geq (a\sqrt{a}+a\sqrt{b}+b\sqrt{a})^2 = (\sqrt{a}(a+b)+a\sqrt{b})^2 =$$
$$= a(a+b)^2+a^2b+2a\sqrt{ab}(a+b)$$

$$(2a^2+b^2)(2a+b) \geq a(a+b)^2+a^2b+2a\sqrt{ab}(a+b) \tag{1}$$

$$(a^2+2b^2)(a+2b) = (a^2+b^2+b^2)(b+a+b) \overset{\text{CBS}}{\geq}$$
$$\geq (a\sqrt{b}+b\sqrt{a}+b\sqrt{b})^2 = ((a+b)\sqrt{b}+b\sqrt{a})^2 =$$
$$= b(a+b)^2+b^2a+2b\sqrt{ab}(a+b)$$

$$(a^2+2b^2)(a+b) \geq b(a+b)^2+b^2a+2b\sqrt{ab}(a+b) \tag{2}$$

By adding $(1);(2)$:

$$LHS \geq (a+b)^3+ab(a+b)+2\sqrt{ab}(a+b)^2 \overset{\text{AM-GM}}{\geq}$$
$$\geq (a+b)^2\cdot 2\sqrt{ab}+ab(a+b)+2\sqrt{ab}(a+b)^2 =$$
$$= \sqrt{ab}(a+b)(2(a+b)+\sqrt{ab}+2(a+b)) =$$
$$= \sqrt{ab}(a+b)(4(a+b)+\sqrt{ab}) \geq 3\sqrt{ab}(a+b)(a+\sqrt{ab}+b)$$
$$4(a+b)+\sqrt{ab} \geq 3(a+b)+3\sqrt{ab}$$
$$a-2\sqrt{ab}+b \geq 0 \Leftrightarrow (\sqrt{a}-\sqrt{b})^2 \geq 0$$

Equality holds for: $a = b$. □

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