PP43630

MIHÁLY BENCZE - ROMANIA

If $x, y, z > 0$ and $xy + yz + zx = 1$ then:

$$\left(\sum_{cyc} \frac{x}{\sqrt{1+x^2}}\right) \cdot \left(\sum_{cyc} \frac{x}{1+x^2}\right) \leq \frac{9\sqrt{3}}{8}$$

Solution by Daniel Sitaru, Claudia Nănuţi.

$$\text{Let be: } x = \cot A; y = \cot B; z = \cot C$$

$$A + B + C = \pi; A, B, C \in \left(0, \frac{\pi}{2}\right)$$

$$\sum_{cyc} \cot A \cot B = 1 \Rightarrow \sum_{cyc} xy = 1$$

$$\left(\sum_{cyc} \frac{x}{\sqrt{1+x^2}}\right) \cdot \left(\sum_{cyc} \frac{x}{1+x^2}\right) =$$

$$= \left(\sum_{cyc} \frac{\cot A}{\sqrt{1+\cot^2 A}}\right) \cdot \left(\sum_{cyc} \frac{\cot A}{1+\cot^2 A}\right) =$$

$$= \left(\sum_{cyc} \frac{\frac{\cos A}{\sin A}}{\frac{1}{\sin A}}\right) \cdot \left(\sum_{cyc} \frac{\frac{\cos A}{\sin A}}{\frac{1}{\sin^2 A}}\right) =$$

$$= \left(\sum_{cyc} \cos A\right) \cdot \left(\sum_{cyc} \cos A \sin A\right) =$$

$$= \frac{1}{2} \cdot \left(1 + \frac{r}{R}\right) \cdot \left(\sum_{cyc} \sin 2A\right) =$$

$$= \frac{1}{2}\left(1 + \frac{r}{R}\right) \cdot 4 \prod_{cyc} \sin A =$$

$$= 2\left(1 + \frac{r}{R}\right) \cdot \frac{4F}{2R^2} = \frac{R+r}{R} \cdot \frac{F}{R^2} =$$

$$= \frac{F(R+r)}{R^3} \overset{\text{EULER}}{\leq} \frac{rs(R + \frac{R}{2})}{R^3} =$$

$$= \frac{3rs}{2R^2} \overset{\text{MITRINOVIC}}{\leq} \frac{3r \cdot \frac{3\sqrt{3}}{2} R}{2R^2} =$$

$$= \frac{9\sqrt{3}r}{4R^2} \overset{\text{EULER}}{\leq} \frac{9\sqrt{3} \cdot \frac{R}{r}}{4R^2} = \frac{9\sqrt{3}}{8}$$

Equality holds for:

$$A = B = C = \frac{\pi}{2} \Rightarrow x = y = z = \frac{\sqrt{3}}{3}$$

□

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