PP44233

MIHÁLY BENCZE - ROMANIA

Let be:

$$A_n = \int_0^1 x^3(1-x^2)^n dx; B_n = \int_0^1 x^5(1-x^2)^n dx$$

a. Find:

$$\lim_{n\to\infty}(n^5 A_n B_n)$$

b. Find:

$$\lim_{n\to\infty}\left(\sum_{k=1}^n A_k\right)\cdot\left(\sum_{k=1}^n B_k\right)$$

Solution by Daniel Sitaru, Claudia Nănuţi.

$$A_n = \int_0^1 x^3(1-x^2)^n dx$$

$$x = \sin y; dx = \cos y dy$$

$$A_n = \int_0^{\frac{\pi}{2}} \sin^3 y \cdot \cos^{2n} y \cdot \cos y dy =$$

$$= \int_0^{\frac{\pi}{2}} \sin y \cdot \sin^2 y \cdot \cos^{2n+1} y dy =$$

$$= \int_0^{\frac{\pi}{2}} \sin y(1-\cos^2 y)\cdot \cos^{2n+1} y dy =$$

$$= \int_0^{\frac{\pi}{2}} \sin y \cos^{2n+1} y dy - \int_0^{\frac{\pi}{2}} \sin y \cos^{2n+3} y dy =$$

$$= -\int_0^{\frac{\pi}{2}} (\cos y)' \cos^{2n+1} y dy + \int_0^{\frac{\pi}{2}} (\cos y)' \cos^{2n+3} dy =$$

$$= -\frac{\cos^{2n+2} y}{2n+2}\Big|_0^{\frac{\pi}{2}} + \frac{\cos^{2n+4}}{2n+4}\Big|_0^{\frac{\pi}{2}} =$$

$$= \frac{1}{2n+2} - \frac{1}{2n+4} = \frac{1}{2(n+1)(n+2)}$$

$$\sum_{k=1}^n A_k = \frac{1}{2}\sum_{k=1}^n \frac{1}{(k+1)(k+2)} =$$

$$= \frac{1}{2}\left(\sum_{k=1}^n \Big(\frac{1}{k+1} - \frac{1}{k+2}\Big)\right) =$$

$$= \frac{1}{2}\Big(\frac{1}{2} - \frac{1}{n+2}\Big) = \frac{n}{4(n+2)}$$

$$B_n = \int_0^1 x^5(1-x^2)^n dx = \int_0^{\frac{\pi}{2}} \sin^5 y \cdot \cos^{2n} y \cdot \cos y dy =$$

$$= \int_0^{\frac{\pi}{2}} \sin y(1-\cos^2 y)^2 \cos^{2n+1} y dy =$$

$$= \int_0^{\frac{\pi}{2}} \sin y(1-2\cos^2 y+\cos^4 y) \cos^{2n+1} y dy =$$

$$= \int_0^{\frac{\pi}{2}} \sin y \cos^{2n+1} y dy - 2\int_0^{\frac{\pi}{2}} \sin y \cos^{2n+3} y dy + \int_0^{\frac{\pi}{2}} \sin y \cos^{2n+5} y dy =$$

$$= -\frac{\cos^{2n+2} y}{2n+2}\Big|_0^{\frac{\pi}{2}} + \frac{2\cos^{2n+4} y}{2n+4}\Big|_0^{\frac{\pi}{2}} - \frac{\cos^{2n+6} y}{2n+6}\Big|_0^{\frac{\pi}{2}} =$$

$$= \frac{1}{2n+2} - \frac{2}{2n+4} + \frac{1}{2n+6} =$$

$$= \frac{1}{2}\Big(\frac{1}{n+1} - \frac{2}{n+2} + \frac{1}{n+3}\Big) = \frac{1}{(n+1)(n+2)(n+3)}$$

$$\sum_{k=1}^{n} B_k = \frac{1}{2}\left(\sum_{k=1}^{n}\Big(\frac{1}{k+1} - \frac{2}{k+2} + \frac{1}{k+3}\Big)\right) =$$

$$= \frac{1}{2} - \frac{2}{3} + \frac{1}{3} + \frac{1}{n+2} - \frac{2}{n+2} + \frac{1}{n+3} =$$

$$= \frac{1}{2} - \frac{1}{3} + \frac{1}{n+3} - \frac{1}{n+2}$$

a. $\lim_{n\to\infty}(n^5 A_n B_n) =$

$$= \lim_{n\to\infty}\Big(n^5 \cdot \frac{1}{2(n+1)(n+2)} \cdot \frac{1}{(n+1)(n+2)(n+3)}\Big) = \frac{1}{2}$$

b. $\lim_{n\to\infty}\Big(\sum_{k=1}^{n} A_k\Big)\Big(\sum_{k=1}^{n} B_k\Big) =$

$$= \lim_{n\to\infty}\Big(\frac{n}{4(n+2)}\Big)\cdot\Big(\frac{1}{6} - \frac{1}{(n+3)(n+2)}\Big) = \frac{1}{4}\cdot\frac{1}{6} = \frac{1}{24}$$

□

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