PP44295

MIHÁLY BENCZE - ROMANIA

In all triangles ABC holds:

$$9r(2R-r)\leq\sum_{cyc}m_a^2\leq 3(2R^2+r^2)$$

Solution by Daniel Sitaru, Claudia Nănuţi.

$$\sum_{cyc}m_a^2=\frac{3}{4}(a^2+b^2+c^2)=\frac{3}{4}\cdot 2(s^2-r^2-4Rr)=$$

$$=\frac{3}{2}(s^2-r^2-4Rr)$$

Remains to prove:

$$9r(2R-r)\overset{(1)}{\leq}\frac{3}{2}(s^2-r^2-4Rr)\overset{(2)}{\leq}3(2R^2+r^2)$$

For (1):

$$9r(2R-r)\leq\frac{3}{2}(s^2-r^2-4Rr)$$
$$2(18Rr-9r^2)\leq 3(s^2-r^2-4Rr)$$
$$36Rr-18r^2\leq 3s^2-3r^2-12Rr$$
$$3s^2\geq 36Rr-18r^2+3r^2+12Rr$$
$$3s^2\geq 48Rr-15r^2$$
$$s^2\geq 16Rr-5r^2 \text{ (Gerretsen)}$$

For (2):

$$\frac{3}{2}(s^2-r^2-4Rr)\leq 3(2R^2+r^2)$$
$$s^2-r^2-4Rr\leq 4R^2+2r^2$$
$$s^2\leq 4R^2+4Rr+3r^2 \text{ (Gerretsen)}$$

□

Mathematics Department, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania

Email address: dansitaru63@yahoo.com