PP44570

MIHÁLY BENCZE - ROMANIA

If $a, b, c > 0; abc = 1$ then:

$$(1+a^4)(1+b^4)(1+c^4) \geq (ab+c)(bc+a)(ca+b)$$

Solution by Daniel Sitaru and Claudia Nănuţi.
We will prove that for $a > 0$:

$$\frac{1+a^4}{1+a^2} \geq \frac{1}{a} \tag{1}$$

$$a + a^5 \geq 1 + a^2 \Leftrightarrow a^5 - a^2 + a - 1 \geq 0$$
$$a^2(a-1)(a^2+a+1) + (a-1) \geq 0 \Leftrightarrow$$
$$(a-1)(a^4+a^2+a+1) \geq 0 \text{ (true)}$$

Analogous:

$$\frac{1+b^4}{1+b^2} \geq \frac{1}{b} \tag{2}$$

$$\frac{1+c^4}{1+c^2} \geq \frac{1}{c} \tag{3}$$

By multiplying $(1); (2); (3)$:

$$\frac{(1+a^4)(1+b^4)(1+c^4)}{(1+a^2)(1+b^2)(1+c^2)} \geq \frac{1}{abc}$$

$$(1+a^4)(1+b^4)(1+c^4) \geq \frac{(1+a^2)(1+b^2)(1+c^2)}{abc} =$$
$$= \left(\frac{1+a^2}{a}\right)\left(\frac{1+b^2}{b}\right)\left(\frac{1+c^2}{c}\right) = \left(\frac{1}{a}+a\right)\left(\frac{1}{b}+b\right)\left(\frac{1}{c}+c\right) =$$
$$= (bc+a)(ac+b)(ab+c)$$

Equality holds for $a = b = c$.

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