PP44591

MIHÁLY BENCZE - ROMANIA

If $a, b, c > 0$ then:

$$1+\frac{1}{abc} \geq 3\Big(\frac{1}{a+b+c}+\frac{1}{a^2+b^2+c^2}\Big)$$

Solution by Daniel Sitaru and Claudia Nănuţi.

$$\text{Denote: } x=\sqrt[3]{abc}>0$$

$$\frac{3}{a+b+c}+\frac{3}{a^2+b^2+c^2} \overset{\text{AM-GM}}{\leq} \frac{3}{3\sqrt[3]{abc}}+\frac{3}{3\sqrt[3]{(abc)^2}}=$$

$$=\frac{1}{x}+\frac{1}{x^2}$$

Remains to prove that:

$$1+\frac{1}{x^3} \geq \frac{1}{x}+\frac{1}{x^2}$$

$$\frac{x^3+1}{x^3} \geq \frac{x+1}{x^2} \Leftrightarrow \frac{x+1}{x^2} \cdot \frac{x^2-x+1}{x} \geq \frac{x+1}{x^2}$$

$$\Leftrightarrow \frac{x^2-x+1}{x} \geq 1 \Leftrightarrow x^2-x+1 \geq x$$

$$\Leftrightarrow x^2-2x+1 \geq 0 \Leftrightarrow (x-1)^2 \geq 0$$

Equality holds for $a=b=c$.

□

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