PP44831

MIHÁLY BENCZE - ROMANIA

In all triangles ABC holds:

$$\frac{6r}{R} \le \sum_{cyc} \frac{(b+c-a)(a-b+c)}{ab} \le \frac{2R-r}{r}$$

Solution by Daniel Sitaru, Claudia Nănuţi.

$$\sum_{cyc} \frac{(b+c-a)(a-b+c)}{ab} = \sum_{cyc} \frac{(2s-2a)(2s-2b)}{ab} =$$

$$= 4\sum_{cyc} \frac{(s-a)(s-b)}{ab} = \frac{4}{abc}\sum_{cyc} c(s-a)(s-b) =$$

$$= \frac{4}{4RF}\sum_{cyc} c(s^2 - s(a+b) + ab) =$$

$$= \frac{1}{RF}\sum_{cyc} c(s^2 - s(2s-c) + ab) =$$

$$= \frac{1}{Rrs}\sum_{cyc}(s^2c + sc^2 + abc) =$$

$$= \frac{1}{Rrs}(-s^2\sum_{cyc} c + s\sum_{cyc} c^2 + 3abc) =$$

$$= \frac{1}{Rrs}(-s^2 \cdot 2s + s \cdot 2(s^2 - r^2 - 4Rr) + 12Rrs) =$$

$$= \frac{1}{Rrs}(-2s^3 + 2s^3 + s(-2r^2 - 8Rr + 12Rr)) =$$

$$= \frac{1}{Rr}(-2r^2 + 4Rr) = \frac{4R-2r}{R}$$

Remains to prove that:

$$\frac{6r}{R} \le \frac{4R-2r}{R} \le \frac{2R-r}{r}$$

$$\frac{6r}{R} \le \frac{4R-2r}{R} \Leftrightarrow 6r \le 4R - 2r \Leftrightarrow 4R \ge 8r$$

$$\Leftrightarrow R \ge 2r \text{ (Euler)}$$

$$\frac{4R-2r}{R} \le \frac{2R-r}{r} \Leftrightarrow 4Rr - 2r^2 \le 2R^2 - Rr \Leftrightarrow$$

$$\Leftrightarrow 2R^2 - 5Rr + 2r^2 \ge 0$$

$$2R^2 - 4Rr - Rr + 2r^2 \ge 0$$

$$2R(R-2r) - r(R-2r) \ge 0$$

$$(R-2r)(2R-r) \ge 0 \text{ (Euler)}$$

Equality holds for $a = b = c$.

□

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