PP44835

MIHÁLY BENCZE - ROMANIA

In all triangles ABC holds:

$$\sum_{cyc}\frac{a^2}{b+c-a}\geq\frac{4s(R-r)}{R}\geq 2s$$

Solution by Daniel Sitaru, Claudia Nănuţi.
We will use a result from the problem PP45141:

$$\sum_{cyc}a^2(s-b)(s-c)=4rs^2(R-r)$$

$$\sum_{cyc}\frac{a^2}{b+c-a}=\sum_{cyc}\frac{a^2}{2s-2a}=\frac{1}{2}\sum_{cyc}\frac{a^2}{s-a}=$$

$$=\frac{1}{2(s-a)(s-b)(s-c)}\sum_{cyc}a^2(s-b)(s-c)=$$

$$=\frac{s}{2s(s-a)(s-b)(s-c)}\cdot 4rs^2(R-r)=$$

$$=\frac{s}{2F^2}\cdot 4rs^2(R-r)=\frac{4rs^3(R-r)}{2r^2s^2}=\frac{2s(R-r)}{r}$$

$$\frac{2s(R-r)}{r}\geq\frac{4s(R-r)}{R}\Leftrightarrow\frac{R-r}{r}\geq\frac{2R-2r}{r}$$

$$\Leftrightarrow R^2-3Rr+2r^2\geq 0\Leftrightarrow(r-2r)(R-r)\geq 0\text{ (True)}$$

$$\frac{4s(R-r)}{R}\geq 2s\Leftrightarrow 2R-r\geq R\Leftrightarrow R\geq 2r\text{ (Euler)}$$

Equality holds for $a=b=c$.

□

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