PP44836

MIHÁLY BENCZE - ROMANIA

In all triangles ABC holds:

$$\frac{4R-5r}{r}\le\sum_{cyc}\frac{(b+c-a)^2}{(a-b+c)(a+b-c)}\le\frac{4R^2-8Rr+3r^2}{r^2}$$

Solution by Daniel Sitaru, Claudia Nănuţi.

$$\sum_{cyc}\frac{(b+c-a)^2}{(a-b+c)(a+b-c)}=\sum_{cyc}\frac{(2s-2a)^2}{(2s-2b)(2s-2c)}=$$

$$=\sum_{cyc}\frac{4(s-a)^2}{4(s-b)(s-c)}=\frac{1}{(s-a)(s-b)(s-c)}\sum_{cyc}(s-a)^3=$$

$$=\frac{s}{s(s-a)(s-b)(s-c)}\cdot\sum_{cyc}(s^3-3s^2a+3sa^2-a^3)=$$

$$=\frac{s}{F^2}(3s^3-3s^2\cdot 2s+3s\cdot 2(s^2-r^2-4Rr)-2s(s^2-r^2-6Rr))=$$

$$=\frac{s^2}{F^2}\cdot(3s^2-6s^2+6s^2-6r^2-24Rr-2s^2+6r^2+12Rr)=$$

$$=\frac{s^2}{r^2s^2}\cdot(s^2-12Rr)=\frac{1}{r^2}(s^2-12Rr)=$$

Remains to prove that:

$$\frac{4R-5r}{r}\le\frac{s^2-12Rr}{r^2}\le\frac{4R^2-8Rr+3r^2}{r^2}$$

$$4Rr-5r^2\le s^2-12Rr\le 4R^2-8Rr+3r^2$$

$$16Rr-5r^2\le s^2\le 4R^2+4Rr+3r^2$$

which is Gerretsen's inequality.
Equality holds for $a=b=c$.

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