PP44837

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In all triangles ABC holds:

$$\frac{9}{2s} \le \sum_{cyc} \frac{a}{(a-b+c)(a+b-c)} \le \frac{9R}{4sr}$$

Solution by Daniel Sitaru, Claudia Nănuţi.

$$\sum_{cyc} \frac{a}{(a-b+c)(a+b-c)} = \sum_{cyc} \frac{a}{(2s-2b)(2s-2c)} =$$

$$= \frac{1}{4}\sum_{cyc} \frac{a}{(s-b)(s-c)} = \frac{1}{4(s-a)(s-b)(s-c)}\sum_{cyc} a(s-a) =$$

$$= \frac{s}{4s(s-a)(s-b)(s-c)} \cdot \left(s\sum_{cyc} a - \sum_{cyc} a^2\right) =$$

$$= \frac{s}{4F^2}(s \cdot 2s - 2s^2 + 2r^2 + 8Rr) = \frac{s}{4r^2s^2}(2r^2 + 8Rr) =$$

$$= \frac{1}{4r^2s} \cdot 2r(r+4R) = \frac{r+4R}{2rs}$$

Remains to prove that:

$$\frac{9}{2s} \le \frac{r+4R}{2rs} \le \frac{9R}{4sr} \Leftrightarrow \begin{cases} 18r \le 2r + 8R \\ 2r + 8R \le 9R \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 8R \ge 16r \\ R \ge 2r \end{cases} \Leftrightarrow \begin{cases} R \ge 2r \\ R \ge 2r \end{cases}$$

which is Euler's inequality.
Equality holds for $a = b = c$.

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